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LOOPS OF CLUTTERS*.

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Abstract. For a clutter $\mathcal{C}$ on a finite set E, let $f(\mathcal{C})$ be the clutter of maximal subsets of E not containing any member of $\mathcal{C}$. The *loop* $L(\mathcal{C})$ of a clutter $\mathcal{C}$ is the finite sequence of clutters $\mathcal{C}, f(\mathcal{C}), f^2(\mathcal{C}), \ldots, f^t(\mathcal{C})$, where t (the *length* of the loop) is the minimum positive integer with $f^t(\mathcal{C}) = \mathcal{C}$. Our motivation of the study of the loop lies on the fact that when $\mathcal{C}$ is the set of circuits of a matroid, the loop contains other critical information associated with the matroid, e.g., the set of bases $f(\mathcal{C})$ and the set of hyperplanes $f^2(\mathcal{C})$. We investigate various properties of the loops, in particular, the possible lengths for fixed $|E|$, the dualities and the symmetries. Our preliminary investigations indicate that there are many interesting problems on the loops whose resolution may provide us with a new insight into Sperner Theory, Matroid Theory and Extremal Set Theory.

1. Introduction. Let E be a finite set of n elements. A *clutter* or *Sperner family* on E is a family of noncomparable subsets of E, i.e.,

$$C, C' \in \mathcal{C} \quad \text{and} \quad C \neq C' \Rightarrow \quad C \not\subseteq C'.$$

For any clutter $\mathcal{C}$, let $f(\mathcal{C})$ be the clutter of all maximal subsets of E containing no member of $\mathcal{C}$. The operator f is much used in matroid theory: f is the operator to obtain the family of bases from the family of circuits of a matroid, and f is the operator to obtain the family of hyperplanes from the family of bases. The importance of this operator often relies on the fact that it has an inverse. For any clutter $\mathcal{C}$, the inverse $f^{-1}(\mathcal{C})$ is simply the family of all minimal subsets of E contained in no member of C. By virtue of this fact one can define matroids in terms of circuits, bases or hyperplanes, or even their complementary sets cohyperplanes, cobases or cocircuits.

In this paper we study the operator f from the following point of view. For any given $\mathcal{C}$, we consider the sequence

$$\mathcal{C},\ f(\mathcal{C}), f^2(\mathcal{C}),\ f^3(\mathcal{C}), \ldots$$

Because f has an inverse and there are finitely many clutters on E, this sequence must come back to the original clutter $\mathcal{C}$ to form the loop:

$$L = L(\mathcal{C}) = \mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2, \ldots,\ \mathcal{C}_{t-1},\ \mathcal{C}_t = \mathcal{C},$$

where $C_i = f^i(\mathcal{C})$ and $t = t(L) > 1$ is the smallest number with $f^t(\mathcal{C}) = \mathcal{C}$, called the *length* of L. Whenever we do not have to specify the first clutter (that is almost always the case in the present paper), we consider the loop L as a cyclically ordered

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set $\{\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_{t-1}\}$. So the cardinality $|L|$ of L is its length $t(L)$, and the equality $L = L'$ will be used to mean that the loops L and L' are equal as cyclically ordered sets.

If $\mathcal{C}$ is the family of circuits of a matroid (or a graph), $\mathcal{C}_1$ is the family of bases (spanning trees) and $\mathcal{C}_2$ is the family of hyperplanes (complements of minimal cutsets). The families $\mathcal{C}_3, \mathcal{C}_4, \ldots, \mathcal{C}_{t-1}$ have never been studied in spite of the fact that they have exactly the same information as the circuits, the bases and the hyperplanes.

The realizable lengths of loops have been studied by Duchet [Du], Brouwer and Shrijver [B-So], and Brouwer [B]. Some related research considering similar loops or operators are [E–Fu], [S], [V].

The main purpose of the present paper is to study two fundamental problems on loops of clutters:

Problem A. For fixed n, (or more generally, a given class of clutters), determine the realizable lengths of loops, in particular the minimum and maximum length.

Problem B. For a given operator h (different from f) associating a clutter to some clutter, find the relationship between $L(\mathcal{C})$ and $L(h(\mathcal{C}))$.

Problem A was first investigated by Duchet [Du] for the particular case of length 2, and more generally by Brouwer and Shrijver [B-Sc]. By an exhaustive search one can easily check that the set of realizable lengths of loops is $\{3\}$ for $n = 1$, $\{2,4\}$ for $n = 2$, $\{5\}$ for $n = 3$, $\{2, 3, 6\}$ for $n = 4$. It was shown in [B-Sc] that the set of realizable lengths for $n = 5$ is $\{2, 3, 7, 16, 27\}$. On the other hand, in general, one can easily verify that the length $n + 2$ is always realizable for any n (e.g., the loop of the clutter $\{E\}$), and that the minimum length is greater than 1. Duchet [Du] proved that for every even n, there always exists a loop of length 2. Separately, Brouwer [B] proved that a loop of length 2 exists for every $n \neq 1, 3$. This settled the problem of determining the minimum lengths for each n. The maximum length appears to be difficult to determine. By computer experiments, we found that the maximum length is at least 1032 for $n = 6$ and 3791 for $n = 7$.

For Problem B, we investigate essentially two choices of h. The first one is the duality operator $^-$:

$$\overline{\mathcal{C}} = \{\overline{C} : C \in \mathcal{C}\},$$

where $\overline{C} = E \backslash C$. It turns out that $L(\mathcal{C})$ and $L(\overline{\mathcal{C}})$ have the same length t, and there is a natural one-to-one correspondence between $L(\mathcal{C})$ and $L(\overline{\mathcal{C}})$:

$$\overline{\mathcal{C}}_i = \mathcal{D}_{t-i} \quad \text{for all } i,$$

where $\mathcal{D} = \overline{\mathcal{C}}$. The implication is obvious: the dual loop is obtained from the original loop $L(\mathcal{C})$ by taking the dual of each clutter $\mathcal{C}_i$ and reversing the order. Consequently, for any loop L, the loop $L(\overline{\mathcal{C}})$ does not depend on the choice of $\mathcal{C} \in L$, and hence there is a uniquely defined loop, the ***dual*** loop $\overline{L}$ of L. Clearly the dual pair $L, \overline{L}$ of loops have the same sum of sizes of clutters:

$$\|L\| = \|\overline{L}\|$$

where $\|L\| = \Sigma\{|\mathcal{C}| : \mathcal{C} \in L\}$. It should be noted that for the "blocker" operator $b = \overline{f}$ (see Section 2 for definition), $L(\overline{\mathcal{C}}) = L(b(\mathcal{C}))$.

The second operator is what we call the reflection operator. This operator associates each k-uniform clutter with another k-uniform clutter:

$$\widetilde{\mathcal{C}} = \{D \subseteq E : |D| = k \quad \text{and} \quad D \notin \mathcal{C}\}.$$

Like the duality operator, this operator is idempotent: $\widetilde{\widetilde{\mathcal{C}}} = \mathcal{C}$. However the analysis of the relationship between $L(\mathcal{C})$ and $L(\widetilde{\mathcal{C}})$ appears to be much more difficult. For all of the uniform clutters $\mathcal{C}$ we investigated, the following surprising properties hold:

(1.1) (*a*) for any uniform clutter $\mathcal{D}$,
$D \in L(\mathcal{C})$ iff $\widetilde{\mathcal{D}} \in L(\widetilde{\mathcal{C}})$;
(*b*) $L(\mathcal{C})$ and $L(\widetilde{\mathcal{C}})$ have the same length;
(*c*) $\|L(\mathcal{C})\| = \|L(\widetilde{\mathcal{C}})\|$.

We conjecture that these properties hold for every uniform clutter $\mathcal{C}$ (the *reflection conjecture*). The property (a) is important because this implies that the pair $L(\mathcal{C})$ and $L(\widetilde{\mathcal{C}})$ do not depend on the choice of $\mathcal{C}$: for any loop L containing a uniform clutter, there exists a unique loop $\widetilde{L}$, the *reflection* loop, satisfying $\widetilde{L} = L(\widetilde{\mathcal{D}})$ for all uniform clutters $\mathcal{D} \in L$. It should be remarked that the properties (b) and (c) seem to come from some nontrivial reasons, since the reflection pair L and $\widetilde{L}$ can have very different characteristics. For example, the distributions of clutter sizes $(|\mathcal{D}| : \mathcal{D} \in L)$ and $(|\mathcal{D}| : \mathcal{D} \in \widetilde{L})$ can be completely different. In this paper we prove this conjecture for many special classes of clutters and, in particular, for all singleton clutters $\{F\}, F \subseteq E$.

Finally, we investigate "symmetry" of loops. The notion of symmetry comes from our observation that many loops L satisfy some of the following strong properties:

(1.2) (*a*) L is self-dual: $L = \overline{L}$;
(*b*) L is self-reflective: $L = \widetilde{L}$;
(*c*) L is self-dual-reflective: $\widetilde{L} = \overline{L}$;

or, more generally,

(1.3) (*a*) L p-self-dual: $L = \overline{L}^p$;
(*b*) L p-self-reflective: $L = \widetilde{L}^p$;
(*c*) L p-self-dual-reflective: $\widetilde{L} = \overline{L}^p$;

for some permutation p of E (where L^p is the image of L under the action of the permutation p). We will show how these properties restrict certain structural properties of loops. For instance, having either the (p−) self-reflection or the (p−)

self-dual-reflection immediately implies that the loop satisfies the reflection conjecture (1.1).

Finally, we conjecture that for a fixed n the average size of clutter $\|L\| \,/\, |L|$ in any loop L is at least $2^n/(n+2)$, which is attained by the *complete uniform loop* $L(\{E\})$. Also, we conjecture that the average size is at most $\binom{n}{\lfloor \frac{n}{2} \rfloor}/2$ for all $n \geq 4$, which is attained by the Brouwer-Duchet loop of length 2. These conjectures are true for all classes of loops for which we could derive an explicit expression of the average size and for many randomly generated loops on few elements. If these conjectures are true, then these expressions provide us with a lower bound and an upper bound of the (universal) average size of clutters on E, which are trivially computable.

For readers unfamiliar with Matroid Theory or Sperner Theory, good references are [W], and [En–G], [G–K].

2. Clutter Operators. Let E be a finite set of cardinality n. For any set $\mathcal{D}$ of subsets of E, let $\min(\mathcal{D})$ and $\max(\mathcal{D})$ denote the family of minimal and maximal sets in $\mathcal{D}$, respectively. For any clutter $\mathcal{C}$ on E, let

$$f(\mathcal{C}) = \max\{D \subseteq E : C \not\subseteq D \quad \text{for all} \quad C \in \mathcal{C}\},$$
$$g(\mathcal{C}) = \min\{D \subseteq E : C \not\subseteq D \quad \text{for all} \quad C \in \mathcal{C}\}.$$

The first result is very elementary but important.

LEMMA 2.1 ([D-F1]). *g is the inverse of f.*

Proof. We will show $g(f(\mathcal{C})) = \mathcal{C}$. If $C \in \mathcal{C}$ then there exists $C' \in g(f(\mathcal{C}))$ with $C' \subseteq C$ by the definition of f and g. Next we take any $C' \in g(f(\mathcal{C}))$. Suppose $C \not\subseteq C'$ for all $C \in \mathcal{C}$. Then, by the definition of f, there exists $D \in f(\mathcal{C})$ such that $C' \subseteq D$, contradicting $C' \in g(f(C))$. Thus there exists $C \in \mathcal{C}$ with $C \subseteq C'$. Since $g(f(\mathcal{C})$ and $\mathcal{C}$ are clutters, they are equal. □

We introduce two operators of clutters closely related to f and g. For a clutter $\mathcal{C}$, let

$$a(\mathcal{C}) = \max\{D \subseteq E : \quad D \cup C \neq E \quad \text{for all} \quad C \in \mathcal{C}\},$$
$$b(\mathcal{C}) = \min\{D \subseteq E : \quad D \cap C \neq \phi \quad \text{for all} \quad C \in \mathcal{C}\}.$$

Note that the operator b is often called the *blocker* operator. Also, define

$$\overline{\mathcal{C}} = \{\overline{C} : C \in \mathcal{C}\},$$

which is the clutter of complements $\overline{C} = E \backslash C$. Clearly, we have

(2.1) $$a(\mathcal{C}) = \overline{g}(\mathcal{C}) \quad \text{and} \quad b(\mathcal{C}) = \overline{f}(\mathcal{C}).$$

More interesting relationships can be obtained by using the well-known equality

LEMMA 2.2 ([E–Fu]). $b^2(\mathcal{C}) = \mathcal{C}$.

LEMMA 2.3.

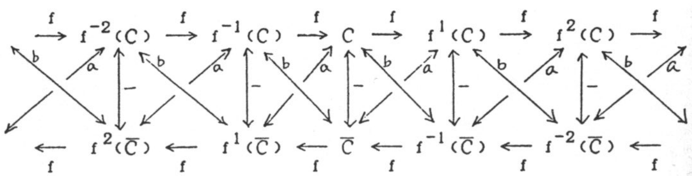
(a) $\overline{f}(\mathcal{C}) = g(\overline{\mathcal{C}})$;
(b) $a(\mathcal{C}) = f(\overline{\mathcal{C}})$;
(c) $b(\mathcal{C}) = g(\overline{\mathcal{C}})$.

Proof. Using Lemma 2.2 and (2.1), we have $f(\overline{f}(\mathcal{C})) = \overline{\mathcal{C}}$. Applying f^{-1} to both sides, we obtain (a) $\overline{f}(\mathcal{C}) = g(\overline{\mathcal{C}})$. The equalities (b) and (c) immediately follow from (2.1) and (a). □

COROLLARY 2.4. $a^2(\mathcal{C}) = \mathcal{C}$.

Proof. Using Lemma 2.3(b) and Lemma 2.2, $a(a(\mathcal{C})) = a(f(\overline{\mathcal{C}}))f(\overline{f}(\overline{\mathcal{C}})) = \mathcal{C}$. □

The following diagram shows the relations among the five operators $f, g, a, b, ^{-}$. Observe that in the diagram there are two loops, the loop $L = L(\mathcal{C})$ of $\mathcal{C}$ and the *dual* loop $\overline{L} = L(\overline{\mathcal{C}})$, and the operators a, b and $^{-}$ send a clutter in one loop to another in the other (its dual) loop while f and $g = f^{-1}$ send a clutter to another in the same loop.



One can easily show:

PROPOSITION 2.5. *Let $\mathcal{C}$ be a clutter on E, and let $L = L(\mathcal{C})$ and $\overline{L} = L(\overline{\mathcal{C}})$. Then*

(a) *L and $\overline{L}$ have the same length;*
(b) *$\mathcal{C}_i = \overline{\mathcal{D}}_{t-i}$ for all i, where $\mathcal{D} = \overline{\mathcal{C}}$;*
(c) *L and $\overline{L}$ have the same sum of sizes: $\|L\| = \|\overline{L}\|$.*

A clutter is called *maximal* if it is not extendable to a larger clutter, i.e., for each $D \subseteq E$ there is $C \in \mathcal{C}$ such that either $D \subseteq C$ or $C \subseteq D$. Let

$$\begin{aligned} f'(\mathcal{C}) &= \{D \in f(\mathcal{C}) : D \not\subseteq C \quad \text{for all} \quad C \in \mathcal{C}\} \\ f''(\mathcal{C}) &= \{D \in f(\mathcal{C}) : D \subseteq C \quad \text{for some} \quad C \in \mathcal{C}\} \\ g'(\mathcal{C}) &= \{D \in g(\mathcal{C}) : D \not\subseteq C \quad \text{for all} \quad C \in \mathcal{C}\} \\ g''(\mathcal{C}) &= \{D \in g(\mathcal{C}) : D \subseteq C \quad \text{for some} \quad C \in \mathcal{C}\}. \end{aligned}$$

Thus, $f(\mathcal{C}) = f'(\mathcal{C}) \cup f''(\mathcal{C})$ and $g(\mathcal{C}) = g'(\mathcal{C}) \cup g''(\mathcal{C})$. Actually, $f''(\mathcal{C})$ consists of sets of form $C \backslash \{a\}$ where $a \in C \in \mathcal{C}$. Clearly,

LEMMA 2.6. *$\mathcal{C} \cup f'(\mathcal{C})$ and $\mathcal{C} \cup g'(\mathcal{C})$ are maximal clutters.*

Using Lemma 2.3, we can easily show

LEMMA 2.7.

(a) $\overline{f}'(\mathcal{C}) = g'(\overline{\mathcal{C}})$;
(b) $\overline{f}''(\mathcal{C}) = g''(\overline{\mathcal{C}})$.

PROPOSITION 2.8. *The following statements are equivalent:*

(a) $\mathcal{C}$ *maximal;*	*(d)* $\overline{\mathcal{C}}$ *is maximal;*
(b) $f'(\mathcal{C}) = \emptyset$;	*(e)* $f'(\overline{\mathcal{C}}) = \emptyset$;
(c) $g'(\mathcal{C}) = \emptyset$;	*(f)* $g'(\overline{\mathcal{C}}) = \emptyset$.

Proof. Since (a) and (d) are clearly equivalent, by Lemma 2.7, it is left to show the equivalence of (a) and (b). Assume (a) and take any $D \in f(\mathcal{C})$. Since $\mathcal{C}$ is maximal, there is $C \in \mathcal{C}$ such that either $D \subseteq C$ or $C \subseteq D$, but the latter cannot hold, because $D \in f(\mathcal{C})$. Thus $D \in f''(\mathcal{C})$, and (b) holds. Now suppose $\mathcal{C}$ is not maximal. Then there is $D \subseteq E$ such that $\mathcal{C} \cup \{D\}$ is a clutter. It follows that there is $D' \supseteq D$ such that $D' \in f(\mathcal{C})$. Clearly $D' \in f'(\mathcal{C})$ and (b) is false. □

It should be noted that an analog of Proposition 2.8 for f'', g'' does not hold: $f''(\overline{\mathcal{C}}) = \emptyset$ is equivalent to $g''(\overline{\mathcal{C}}) = \emptyset$ but not equivalent to $f''(\overline{\mathcal{C}}) = \emptyset$ (i.e., $\mathcal{C} \cup f(\mathcal{C})$ and $\mathcal{C} \cup f(\mathcal{C})$ and $\mathcal{C} \cup g(\mathcal{C})$ do not necessarily form clutters simultaneously).

3. Theorem of Alternatives on Loops.

Edmonds and Fulkerson [E–F] observed that the following alternative property holds for clutters:

PROPOSITION 3.1. *Let $\mathcal{C}$ be a clutter on E. Then for any $F \subseteq E$, exactly one of the following alternatives holds:*

(a) *$\exists\, A \in \mathcal{C}$ such that $A \subseteq F$;*
(b) *$\exists\, B \in f(\mathcal{C})$ such that $F \subseteq B$.*

Proof. Immediate from the definition of f. □

Recall that $\mathcal{C}_j = f^j(\mathcal{C})$ for any clutter $\mathcal{C}$ and each j.

PROPOSITION 3.2. *Let $\mathcal{C}$ be a clutter on E. Fix indices i, j with $i < j$, and $F \subseteq E$. Then if $F \notin \mathcal{C}_k$ for $k = i+1, \ldots, j-1$ then at least one of the following alternative holds:*

(a_i) *$\exists\, A \in \mathcal{C}_j$ such that $A \subseteq F$;*
(b_j) *$\exists\, B \in \mathcal{C}_j$ such that $F \subseteq B$.*

Proof. By Proposition 3.1, we know that one of the statements always holds:

(a_{k-1}) $\exists\, A \in \mathcal{C}_{k-1}$ such that $A \subseteq F$;
(b_k) $\exists\, B \in \mathcal{C}_k$ such that $F \subseteq B$.

Similarly, one of the following statements holds:

(a_k) $\exists\, A \in \mathcal{C}_k$ such that $A \subseteq F$;
(b_{k+1}) $\exists\, B \in \mathcal{C}_{k+1}$ such that $F \subseteq B$.

Since $\mathcal{C}_k$ is a clutter, both (b_k) and (a_k) hold iff $F \in \mathcal{C}_k$. So if $F \notin \mathcal{C}_k$ then at least one of (a_{k-1}) and (b_{k+1}) hold. By inductive argument the result follows. □

Proposition 3.2 has some interesting corollaries. For instance, if we set $j = t+i$ (i.e., $\mathcal{C}_j = \mathcal{C}_i$), where t is the length of the loop $L(\mathcal{C})$, then we immediately obtain:

COROLLARY 3.3. *Let $\mathcal{C}$ be a clutter on E and let $F \subseteq E$. If F is comparable with no member of $\mathcal{C}$ then $F \in \mathcal{C}_j$ for some j.*

The next two corollaries follow naturally.

COROLLARY 3.4. *Let $\mathcal{C}$ be a k-uniform clutter on E. Then every k-subset of E appears (in some clutter) in the loop $L(\mathcal{C})$.*

COROLLARY 3.5. *Every r-subset of E appears in the loop of any matroid of rank r.*

Corollary 3.3 gives a sufficient condition for a subset of E to appear in the loop $L(\mathcal{C})$. It is easy to see that this is not a necessary condition. However, we have:

PROPOSITION 3.6. *Let $\mathcal{C}$ be a clutter on E. Then a subset F of E is a member of $\mathcal{C}_k$ for some k if and only if there exists j such that no member of $\mathcal{C}_j$ is comparable with F.*

Proof. It is enough to show the "only if" part. Without loss of generality we may assume $F \in \mathcal{C}_0$. Suppose that for each j there exists some member of $\mathcal{C}_j$ which is comparable with F. This implies that there is $C \in \mathcal{C}_1$ such that $C \subseteq F$. So there is $C' \in \mathcal{C}_2$ such that $C' \subseteq F$, and so on. Consequently some $C'' \in \mathcal{C}_{t-1}$ must be contained in F, contradicting the assumption. □

4. Group of Permutations Preserving a Clutter. For a given clutter $\mathcal{C}$ on E, define $G = G(\mathcal{C}) = \mathrm{Aut}\,\mathcal{C}$, the group of all permutations of E preserving $\mathcal{C}$.

Obviously, we have $G(\mathcal{C}) = G(\overline{\mathcal{C}})$. The following is less trivial.

LEMMA 4.1. $G(\mathcal{C}) = G(f(\mathcal{C}))$.

Proof. Since the clutters $f^i(\mathcal{C})$ form a loop, it is enough to show $G(f(\mathcal{C}) \supseteq G(\mathcal{C})$. We must show that $g(b) \in f(\mathcal{C})$ for any fixed $b \in f(\mathcal{C})$ and $g \in G\backslash\langle 1\rangle$ where $G = G(\mathcal{C})$. In fact, otherwise, we must have either (i) $g(b) \supseteq a$ for some $a \in \mathcal{C}$

(impossible since then $b \supseteq g^{-1}(a) \in \mathcal{C}$) or (ii) $g(b) \subsetneq b'$ for some $b' \in f(\mathcal{C})$. We prove the impossibility of case (ii) by induction for decreasing $|b|$. For maximal $|b|$, say m, it is impossible since $|b'| \leq |b|$. For $|b| = m-1$ (if any), $g(b) \subsetneq b'$ would imply $b \subsetneq g^{-1}(b')$, contradicting $f(\mathcal{C})$ being a clutter, for $g^{-1}(b') \in f(\mathcal{C})$ by induction. □

COROLLARY 4.2. $G(\mathcal{C}) = G(g(\mathcal{C})) = G(a(\mathcal{C})) = G(b(\mathcal{C}))$.

Proof. Follows from Lemmas 2.3, 4.1 and the equality $G(\mathcal{C}) = G(\overline{\mathcal{C}})$. □

Lemma 4.1 implies that every clutter in a loop has the same group G. But it is possible to have different loop with the same G. For example, $\mathcal{C}$ and $\overline{\mathcal{C}}$ may produce different loops, but $G(\mathcal{C}) = G(\overline{\mathcal{C}})$. Other examples will be given in the next section (Example 5.3).

We denote by S_E the group all permutations of E. For a subgroup G of $S_{E'}$ a clutter $\mathcal{C}$ is called *G-clutter* if $G(\mathcal{C}) = G$, and the loop L associated with G-clutter is called a *G-loop*.

It is not difficult to see that there are some subgroups G not admitting a G-clutter. For example, for $E = \{1,2,3\}$, the trivial group $G = \langle 1\rangle$ does not admit a G-clutter (i.e., every clutter on E is preserved by some nontrivial permutation).

Problem 4.3. Characterize subgroups $G \leq S_E$ admitting a G-clutter.

More general treatment of automorphism groups and of symmetries of loops will be in [D–F2].

5. The Reflection Conjecture and Examples. In this section we shall introduce many interesting examples of loops which can be analyzed. In particular, we emphasize showing what types of symmetry each loop has, and showing the validity of the reflection conjecture (1.1) for each loop. Let us recall some important definitions.

The reflection operator $\sim$ is the mapping which associates each k-uniform clutter with another k-uniform clutter:

$$\widetilde{\mathcal{C}} = \{D \subseteq E : |D| = k \quad \text{and} \quad D \notin \mathcal{C}\}.$$

For each loop L containing a uniform clutter, if $L(\widetilde{\mathcal{D}})$ does not depend on the choice of uniform clutters $\mathcal{D} \in L$, then we call this unique loop the *reflection loop* of L and denote it by $\widetilde{L}$.

The reflection conjecture says that for each loop L containing a uniform clutter,

(a) the reflection loop $\widetilde{L}$ exists;
(b) $|L| = |\widetilde{L}|$ (the same length);
(c) $\|L\| = \|\widetilde{L}\|$ (the same sum of sizes).

Recall that we defined the following three notions of symmetries:

(a) *self-dual*: $L = \overline{L}$;
(b) *self-reflective*: $L = \widetilde{L}$;
(c) *self-dual-reflective*: $\widetilde{L} = \overline{L}$;

and the following more general notions:

(a) *p-self dual:* $L = \overline{L}^p$;
(b) *p-self-reflective:* $L = \widetilde{L}^p$;
(c) *p-self-dual-reflective:* $\widetilde{L} = \overline{L}^p$;

where p is a permutation p of E (and L^p is the image of L under the action of the permutation p). So we have trivially

PROPOSITION 5.1. *If a loop L is either $(p-)$ self-reflection or $(p-)$ self-dual-reflection, then it satisfies the reflection conjecture (1.1).*

Example 5.2. The complete uniform loop

We call a clutter $\mathcal{C}$ ***uniform*** or ***k-uniform*** if every member C of $\mathcal{C}$ has the same cardinality k. The k-uniform clutter $(E;k) := \{C \subseteq E: \ |C| = k\}$ is called the ***complete k-uniform*** clutter, for $k \geq 0$. For convenience, we define $(E;-1) = \emptyset$, the empty clutter. Note that the empty clutter $(E;-1)$ is different from $(E;0) = \{\emptyset\}$, and can be considered as a k-uniform clutter for any $0 \leq k \leq n$.

PROPOSITION 5.2.1. *The S_E-clutters are exactly the complete uniform clutters $(E;k)$, $-1 \leq k \leq n$, and form the loop of length $n+2$:*

$$L((E;n)) = (E;n),\ (E;n-1), \ldots, (E;1),\ (E;0),\ (E;-1),\ (E;n),$$

*which will be called the **complete uniform loop**.*

Since every subset of E appears exactly once in the loop, the sum of sizes $\|L((E;n))\| = 2^n$, and the average size of clutters in the loop is $2^n/(n+2)$. Observing that $\overline{(E;k)} = (E;n-k)$ and $\widetilde{(E;k)} = (E;-1)$ for all $0 \leq k \leq n$, we see that the complete uniform loop is both self-dual and self-reflection, hence the reflection conjecture is satisfied.

Example 5.3. Two $S_{E-\{1\}}$-loops

For $n \geq 2$, let

$$\mathcal{C} = \{\{1\}\}$$
$$\mathcal{C}' = \{\{1\}, E - \{1\}\}.$$

While the clutters $\mathcal{C}$ and $\mathcal{C}'$ (on E) have the same automorphism group $S_{E-\{1\}}$, they produce different loops. Both loops $L(\mathcal{C})$ and $L(\mathcal{C}')$ have the same length $(n+2)$, the same sum of sizes 2^n, and have exactly four uniform clutters. The first loop $L(\mathcal{C})$ contains two 1-uniform clutters and two $(n-1)$-uniform clutters. On the other hand, the second loop $L(\mathcal{C}')$ contains two 2-uniform clutters and two $(n-2)$-uniform clutters. Both loops are self-dual and self-reflective, and hence satisfy the reflection conjecture. These facts can be easily verified using some technique for splittable clutters developed in the next section.

Example 5.4. Loops of rigid clutters

A clutter $\mathcal{C}$ (or its loop) is ***rigid*** if $G(\mathcal{C}) = \langle 1 \rangle$. The simplest rigid clutter is $\{1\}$ on $E = \{1,2\}$, and its loop $\{1\}$, $\{2\}$ is obviously self-dual and self-reflective. There are no rigid clutters on E with $n = 3$ or 4. For $n = 5$, we have the following example.

Let $\mathcal{C} = \{13, 14, 23, 345\}$. Here we employ a simplified notation for clutters: 13 and 345, say, mean the sets $\{1, 3\}$ and $\{3, 4, 5\}$. In the sequel, whenever there are no ambiguities, we use this notation. Then $\mathcal{C}$ is a rigid loop on $\{1, 2, 3, 4, 5\}$. The loop $L(\mathcal{C})$ has length 7 and contains no uniform clutters.

Example 5.5. Brouwer-Duchet loops [Du], [B]

Let $|E| = n = 2k$. Let

$$\mathcal{C}_i = \{C \subseteq E : |C| = k \quad \text{and} \quad |\{1, 2, \ldots k\} \cap C| = i \pmod 2\}$$

for $i = 0, 1$. Then it is easy to see that $L(\mathcal{C}) = \{\mathcal{C}_0, \mathcal{C}_1\}$. The average size of clutters is $\binom{n}{\frac{n}{2}}/2$.

When k is odd, $\mathcal{C}_0 = \overline{\mathcal{C}}_1$, and when k is even $\mathcal{C}_0 = \overline{\mathcal{C}}_0$. This implies that the loop is self-dual. Moreover, we have $\mathcal{C}_0 = \widetilde{\mathcal{C}}_1$, hence it is self-reflective and thus satisfies the reflection conjecture.

Brouwer [B] constructed loops of length 2 for every odd $n \neq 1, 3$ also but here we omit the description of his construction because it is a little complicated.

Example 5.6. Loops of stars

For $1 \leq s \leq k \leq n$, let

$$\mathrm{Star}(s, k, n) = \{C \subseteq E: \ \{1, 2, \ldots, s\} \in C, \quad |C| = k\}.$$

Set $\mathcal{C} = \mathrm{Star}(1, k, 2k)$. Then one can verify that the length of the loop $L(\mathcal{C})$ is $k+1$, the sum of sizes is 2^{2k+1}, and the average size of clutters is $2^n/(n+2)$. This loop contains exactly two uniform clutters, $\mathcal{C}$ and $\overline{\mathcal{C}} \equiv \widetilde{\mathcal{C}}$, hence it is self-dual and self-reflective.

Example 5.7. Loops of the Fano matroid and Steiner systems

Let $\mathcal{C}$ be the set of circuits of the Fano matroid M on $E = \{1, 2, \ldots, 7\}$, i.e., $\mathcal{C} = \{126, 135, 147, 234, 257, 367, 456, 1237, 1245, 1347, 1567, 2356, 2467, 3457\}$. Then the loop $L(\mathcal{C})$ is

$\mathcal{C}_0 =$ the set of circuits $= \mathcal{C}_2 \cup \mathcal{C}_3$
$\mathcal{C}_1 =$ the set of bases $= S_4(2, 3, 7)$
$\mathcal{C}_2 =$ the set of of hyperplanes (lines) $= S(2, 3, 7) = \widetilde{\mathcal{C}}_1$
$\mathcal{C}_3 =$ the set of cocircuits $= S_2(2, 4, 7)$
$\mathcal{C}_4 =$ the set of cobases $= S_8(2, 4, 7) = \widetilde{\mathcal{C}}_3$
$\mathcal{C}_5 =$ the set of cohyperplanes $= \mathcal{C}_0$.

Clearly it is self-dual and its length is 5. All known examples of loops for $n = 7$ have longer lengths. Except for $\mathcal{C}_0$, all clutters are uniform and $\mathcal{C}_1 = \widetilde{\mathcal{C}}_2$ and $\mathcal{C}_3 = \widetilde{\mathcal{C}}_4$. Thus it is self-reflective as well.

More general loops with self-reflective property can be obtained as follows. For $0 \le s < k \le n$, a clutter $\mathcal{C}$ is called an s-design $S_\lambda(s,k,n)$ if it is a k-uniform clutter such that every s-subset of E is contained in exactly λ members of $\mathcal{C}$. In particular, an s-design $S_1(s,k,n)$, denoted by $S(s,k,n)$, is called a *Steiner system*, and an s-design $S(s,3,n)$ is called a Steiner *triple* system. The (set of hyperplanes $\mathcal{C}_2$ of the) Fano matroid is the Steiner triple system $S(2,3,7)$.

PROPOSITION 5.7.1. *Let $\mathcal{C}$ be a design $S_\lambda(s,k,n)$. Then $s+1 \le |D| \le s+\lambda$ for all $D \in f^{-1}(\mathcal{C})$. In particular, $f^{-1}(\mathcal{C})$ is $(s+1)$-uniform if $\mathcal{C}$ is a Steiner system (i.e. $\lambda = 1$) and $f^{-1}(\mathcal{C}) = \widetilde{\mathcal{C}}$ if, in addition, $s = k-1$.*

Proof. Let $D \in f^{-1}(\mathcal{C})$. By Lemma 2.1, D is a minimal set not contained in any member of $\mathcal{C}$. Suppose $|D| > s+\lambda$ and let $a_1, a_2, \dots, a_{\lambda+1}$ be distinct members of D. Consider $\lambda+1$ sets $D_i = D\backslash\{a_i\}$. Each of them is contained in some member of $\mathcal{C}$ (because of minimality of D) but in a different one (because, otherwise, it will contain D). Fix a s-subset T of $D\backslash\{a_1, a_2, \dots, a_{\lambda+1}\}$ (it is possible since $|D| > s+\lambda$ by the assumption). So, we have $\lambda+1$ different members of $\widetilde{\mathcal{C}}$ (blocks) containing s-set T, but $S_\lambda(s,k,n)$ has exactly λ such blocks. □

Both bounds of Proposition 5.7.1 are not best possible. For example, for $\mathcal{C} = S_{12}(2,4,9)$ (respectively, $S_8(2,4,7)$) we have $f^{-1}(\mathcal{C}) = \widetilde{\mathcal{C}}$ and so $3 < |D| = 4 < 14$ (respectively, $3 < |D| = 4 < 10$).

We observe that the loop of the Steiner system $S(2,3,9)$:

$$\mathcal{C} = \{123, 147, 159, 168, 249, 258, 267, 348, 357, 369, 456, 789\}$$

is very similar to that of the Fano matroid: it has length 5, contains four uniform clutters, and is self-reflection:

$$\mathcal{C}_0 = \widetilde{\mathcal{C}}_4 \quad \text{(3-uniform)},$$
$$\mathcal{C}_1 = \widetilde{\mathcal{C}}_2 \quad \text{(4-uniform)}.$$

However, the loop is not self-dual.

There is a following striking analogy among the loops of the affine plane $AG(2,3) = S(2,3,9)$, of the projective plane $PG(2,2) = S(2,3,7)$ (Fano loop), and the loop $L3-2$ (the loop of {1} on {1, 2, 3})

Loop	$\mathcal{C}_0$	$\mathcal{C}_1 = \widetilde{\mathcal{C}}_0$	$\mathcal{C}_2$	$\mathcal{C}_3 = \widetilde{\mathcal{C}}_4$	$\mathcal{C}_4$
$AG(2,3)$	$S_9(2,4,9)$	$S_{12}(2,4,9)$	$\mathcal{C}_0 \cup \mathcal{C}_4$	$S_6(2,3,9)$	$S(2,3,9)$
$PG(2,2)$	$S_2(2,4,7)$	$S_8(2,4,7)$	$\mathcal{C}_0 \cup \mathcal{C}_4$	$S_4(2,3,7)$	$S(2,3,7)$
$L3-2$	{23}	{12, 13}	$\mathcal{C}_0 \cup \mathcal{C}_4$	{2, 3}	{1}

In each of the three cases, the clutter $\mathcal{C}_2$ can be considered as the set of circuits of some matroid, $\mathcal{C}_3$ is the set of bases, $\mathcal{C}_2$ is the set of hyperplanes, and except for $AG(2,3)$, $\mathcal{C}_0$ is the set of cocircuits and $\mathcal{C}_1$ is the set of cobases. All those three loops and the dual to $AG(2,3)$ are loops of length 5 satisfying the "loop equations" $\mathcal{C}_1 = \widetilde{\mathcal{C}}_0$, $\mathcal{C}_3 = \widetilde{\mathcal{C}}_4$, $\mathcal{C}_2 = \mathcal{C}_0 \cup \mathcal{C}_4$. It would be interesting to know if there exist other such loops.

Observe that, in general, if $\mathcal{C} = S_\lambda(s,k,n)$, then $\mathcal{C} = S_{\binom{n-s}{k-s}-\lambda}(s,k,n)$.

Example 5.8. Isometric clutters $\mathcal{C}$, $f(\mathcal{C})$

For any two sets A and B, the Hamming distance $d(A,B)$ between them is defined as the cardinality of the symmetric difference, i.e., $|A \cup B| - |A \cup B|$. We call two clutters $\mathcal{C}$, $\mathcal{C}'$ ***isometric*** if there is a bijection α between them such that for any $A, B \in \mathcal{C}$, $d(A,B) = d(\alpha(A), \alpha(B))$.

(a) Two isometric "open snakes" for $n = 4$

$\mathcal{C}$	$f(\mathcal{C})$
12	1 3
23	1 4
34	2 4

(b) Two isometric "closed snakes" for $n = 5$

$\mathcal{C}$	$f(\mathcal{C})$
12	1 3
23	1 4
34	2 4
45	2 5
1 5	3 5

In both cases (a) and (b), $f(\mathcal{C}) = \widetilde{\mathcal{C}}$ and it consists of all pairs which are not in $\mathcal{C}$, because any 3-set contains a set from $\mathcal{C}$. (Any 3-subset of a 4-set consists of 3 consecutive points in cyclic order. Any 3-subset of a 5-set is either three consecutive points in cyclic order or of form 1 34.) Moreover $f^2(\mathcal{C}) = \mathcal{C}$, i.e., (a) and (b) are loops of length 2, since any 3-set contains a set from $f(\mathcal{C})$. These examples are quite exceptional.

Both snakes above and the Brouwer-Duchet loops (Example 5.6) satisfy the loop equations $\mathcal{C}_0 = \widetilde{\mathcal{C}}_1 = \mathcal{C}_2$. Another example (due to A. Blokhuis) is the closed snake (2 Fano planes) for $n = 7$, $\mathcal{C} = \{124, 235, 346, 457, 561, 672, 713, 764, 653, 542, 431, 327, 216, 175\}$.

Another example of a k-uniform clutter such that $f(\mathcal{C})$ consists of all remaining k-subsets, satisfying $f(\mathcal{C}) = \widetilde{\mathcal{C}}$, is

$$\begin{array}{l} 12\ 4 \\ 12\ \ 5 \\ \ 23\ 5 \\ \ 23\ \ 6 \\ \ \ 34\ 6 \\ 1\ 34 \\ 1\ \ \ 45 \\ \ 2\ \ 45 \\ \ 2\ \ \ \ 56 \\ \ \ 3\ \ \ 56 \end{array}$$

because any 4-subset of a 6-set contains a triplet from this clutter. (Check three cases for 4-sets: 1234, 123 5, 12 45.) But $f^2(\mathcal{C}) \neq \mathcal{C}$ in this case, because it contains 1245, 2356, 1346.

(c) A general example of an isometric $\mathcal{C}$ satisfying $f(\mathcal{C}) = \overline{\mathcal{C}}$

Case 1. $|E| = 2k - 1$

$$\mathcal{C} = (E;\ k), \quad f(\mathcal{C}) = (E;\ k-1), \text{ so } f(\mathcal{C}) = \overline{\mathcal{C}}.$$

Case 2. $|E| = 2k$

$$\mathcal{C} = (E - \{2k\}; k), \quad f(\mathcal{C}) = \overline{\mathcal{C}} \quad .$$

6. Splittable Clutters. For a given clutter $\mathcal{C}$, it is in general very hard to compute $f(\mathcal{C})$: the computational complexity is clearly exponential in the size of $\mathcal{C}$ and n since the output size can be exponential in n. In this section, we shall introduce some classes of clutters for which the computation $f(\mathcal{C})$ is easy.

Let (E_1, E_2) be a given partition of E, and let

$$(p, q) = (E_1; p) \times (E_2; q),$$

i.e., the $(p+q)$-uniform clutter of sets consisting of some p-set from E_1 and some q-set from E_2. A clutter $\mathcal{C}$ is called ***splittable*** if there is a nontrivial partition (E_1, E_2) of E such that

$$\mathcal{C} = (p_1, q_1) \cup (p_2, q_2) \cup \cdots \cup (p_h, q_h),$$

for some p_i's and q_i's. In the expression above, since $\mathcal{C}$ is a clutter, we may suppose

$$s \geq p_1 > p_2 > \cdots > p_h \geq 0 \quad \text{and}$$
$$0 \leq q_1 < q_2 < \cdots < q_h \leq n - s,$$

where $s = |E_1|$. Clearly $h \leq \min\{s, n-s\} + 1$, and hence splittable clutters have a compact representation. The automorphism group Aut $\mathcal{C}$ is $S_{E_1} \times S_{E_2}$.

First we note that some of the examples in the previous section are splittable. For $|E| = n = 2k$, letting $E_1 = \{1, \ldots, k\}$ and $E_2 = \{k+1, \ldots n\}$, the Brouwer-Duchet clutters (Example 5.5) can be written as

$$\mathcal{C}_0 = (k'', k - k'') \cup \cdots \cup (4, k-4) \cup (2, k-2) \cup (0, k) \quad \text{and}$$
$$\mathcal{C}_1 = (k', k - k') \cup \cdots \cup (3, k-3) \cup (1, k-1),$$

where k' and k'' are the largest odd and even number not greater than k, respectively. One can easily see that the two $S_{E-\{1\}}$-clutters (Example 5.3) and the star families Star(s, k, n) (Example 5.6) are also splittable.

PROPOSITION 6.1. *Let (E_1, E_2) be a nontrivial partition of E with $s|E_1| \leq |E_2|$. Then the total number of (E_1, E_2)-splittable clutters is*

$$\sum_{h=1}^{s+1} \binom{s+1}{h} \binom{n-s+1}{h}.$$

PROPOSITION 6.2. *If $\mathcal{C}$ is (E_1, E_2)-splittable then $f(\mathcal{C})$ is also (E_1, E_2)-splittable and*

$$f(\mathcal{C}) = (s, q_1 - 1) \cup (p_1 - 1, q_2 - 1) \cup$$
$$(p_2 - 1, q_3 - 1) \cup \cdots \cup (p_{h-1} - 1, q_h - 1) \cup (p_h - 1, n - s).$$

We omit the proofs as they are quite straightforward. Note that in the expression of $f(\mathcal{C})$ above the first term and/or the last term may be empty, because $q_1 - 1$ and $p_h - 1$ can be negative.

THEOREM 6.3. *The loop $L(\mathcal{C})$ of any 1-uniform clutter $\mathcal{C}$ is self-dual reflective, and hence satisfies the reflection conjecture.*

Proof. Let $\mathcal{C}$ be a 1-uniform clutter. Since the result is true for the complete uniform clutter, we assume $\mathcal{C}$ is not complete. Clearly, $\mathcal{C}$ is splittable and can be written as (1, 0) for some nontrivial partition (E_1, E_2). Now using Proposition 6.2, we have

$$\begin{aligned}
\mathcal{C} = \mathcal{C}_0 &= (1,\ 0) && :\ 1 - \text{uniform clutter} \\
\mathcal{C}_1 &= (0, n-s) && :\ (n-s) - \text{uniform clutter} \\
\mathcal{C}_2 &= (s, n-s-1) && :\ (n-1) - \text{uniform clutter} \\
\mathcal{C}_3 &= (s, n-s-2) \cup (s-1, n-s) \\
\mathcal{C}_4 &= (s, n-s-3) \cup (s-1, n-s-1) \cup (s-2, n-s) \\
&\ \vdots \\
\mathcal{C}_j &= (s, n-s-(j-1)) \cup (s-1, n-s-(j-3)) \cup \cdots \cup (s-(j-2), n-s), \\
&\qquad\qquad (j \leq \min\{n-s+1, s+2\}) \quad .
\end{aligned}$$

Now consider the case $s+2 < n-s+1$ (the cases $s+2 = n-s+1$ and $s+2 > n-s+1$ can be treated similarly). Then

$$\begin{aligned}
\mathcal{C}_{s+2} &= (s, n-2s-1) \cup (s-1, n-2s+1) \cup \cdots \cup (0, n-s) \\
\mathcal{C}_{s+3} &= (s, n-2s-2) \cup (s-1, n-2s) \cup \cdots \cup (0, n-s-1) \\
&\ \vdots \\
\mathcal{C}_{n-s+1} &= (s,0) \cup (s-1,2) \cup \cdots \cup (0, s+1) \\
\mathcal{C}_{n-s+2} &= (s-1,1) \cup \cdots \cup (0,s) \qquad : s-\text{uniform clutter} \\
\mathcal{C}_{n-s+3} &= (s,0) \cup (s-2,1) \cup \cdots \cup (0, s-1) \\
\mathcal{C}_{n-s+4} &= (s-1,0) \cup \ldots U(0, s-2) \\
&\ \vdots \\
\mathcal{C}_{n-s+(s+1)} &= (2,0) \cup (0,1) \\
\mathcal{C}_{n-s+(s+2)} &= (1,0) = \mathcal{C}_0.
\end{aligned}$$

It follows that the loop L has length $n-s+(s+2) = n+2$, and has exactly four uniform clutters, $\mathcal{C}_0 = (1,0)$, $\mathcal{C}_1 = (0, n-s)$, $\mathcal{C}_2 = (s, n-s-1)$, $\mathcal{C}_{n-s+2} = (s-1,1) \cup \cdots \cup (0,s)$. Since

$$\begin{aligned}
\widetilde{\mathcal{C}}_0 &= \overline{\mathcal{C}}_2 \\
\widetilde{\mathcal{C}}_1 &= \overline{\mathcal{C}}_{n-s+2},
\end{aligned}$$

the loops $L(\widetilde{\mathcal{D}})$, $\mathcal{D} = \mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_{n-s+2}$ are all the same, and equal to the dual loop $\overline{L}$. This implies that the reflection loop $\widetilde{L}$ exists and the loop L is self-dual-reflection. The proof is complete. □

As we see in the proof, any singleton clutter $\mathcal{C} = \{F\}, F \subseteq E$, appears (as $\mathcal{C}_1$) in the loop of some 1-uniform clutter. Thus,

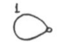
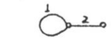
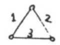
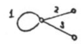
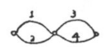
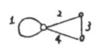
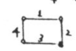
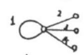
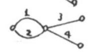
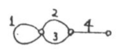
COROLLARY 6.4. *The reflection conjecture (1.1) is true for all singleton clutters.*

7. Catalog of Loops and Some Experimental Results. At present we don't have strong techniques for analyzing loops of clutters. So, it will be helpful for us to employ some enumerative approaches or computer experiments.

7.1. Catalog of loops for $n \leq 4$. First we give a complete catalog of loops for $1 \leq n \leq 4$. For enumeration, of course we identify isomorphic loops. Here we say that two loops L and L' on E are *isomorphic* if there is a permutation p of E such that $L' = L^p$.

Table 1 contains all nonisomorphic loops for $n \leq 4$. For each loop L, we tried to associate a graph whose set of circuits generates L. There is only one loop, the open snake L4-1, for which this is impossible ("nongraphic loop"). One can easily observe that all loops are ($p-$) self-dual and ($p-$)-self-reflection. So the reflection conjecture is true for $n \leq 4$.

TABLE 1. Complete catalogue of loops for $n \leq 4$

Loop	$\lvert L \rvert$	$\lVert L \rVert$	C_0	Symmetry	Comment
(n = 1)					
L1-1	3	$2=2^n$		SD, SR	CU
(n = 2)					
L2-1	2	2		SD, SR	Sp, Ri, Du, St
L2-2	4	$4=2^n$		SD, SR	CU
(n = 3)					
L3-1	5	$8=2^n$		SD, SR	CU
L3-2	5	8		SD, SR	Sp
(n = 4)					
L4-1	2	6	{12, 23, 34}	SD, p-SR, p=(2 3)	open snake
L4-2	2	6		SD, SR	Sp, Du
L4-3	3	8		SD, SR	Sp, St
L4-4	6	$16=2^n$		SD, SR	CU
L4-5	6	16		SD, SR	Sp
L4-6	6	16		p-SD, p-SR, p=(1 3) (2 4)	Sp
L4-7	6	16		p-SD, p-SR, p=(1 4)	

Abbreviations: CU = Complete Uniform, Sp = Splittable, Ri = Rigid
Du = Duchet clutter, St = Star clutter Star(1, k, 2k)
SD = Self-Dual, SR = Self-Reflection

For example for the case $n = 2$ we have two loops

$$L2-1 = \{12\},\quad \{1,2\},\quad \{\emptyset\},\quad \emptyset \quad \text{and} \quad L2-2 = \{1\},\ \{2\}$$

In the case $n = 3$, there are exactly 19 clutters, including 2 "nonproper" clutters: $\{123\}$, $\{\emptyset\}$. This set of 19 clutters is partitioned by the following 4 loops of clutters:

$$L3-1 = \{123\},\quad \{12,23,13\}, \{1,2,3\},\quad \{\emptyset\},\ \emptyset$$

and three loops of the type

$$L3-2 = \{1\},\{23\},\quad \{12,13\},\quad \{1,23\},\quad \{2,3\}.$$

The maximal (i.e., nonextendable) clutters are the four clutters of the first loop and the clutters

$$\{1,23\},\quad \{2,13\},\quad \{3,12\}$$

i.e., one from each of the three remaining loops.

We should also note that the loop L3–2 is isomorphic to the loop of the Fano matroid, $L(S(2,3,7))$, up to the equivalence on subsets of E induced by G-orbits on 2^E, where G is the corresponding automorphism groups $\langle(23)\rangle$ and $\langle(145)(276),(26)(45)\rangle$ (details in [D-F2]). Each of these 3 loops has form $\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_0 \cup \mathcal{C}_1, \widetilde{\mathcal{C}}_0$ with $\mathcal{C}_0 = \{1\}, S(2,3,7),\ S(2,3,9)$ and $\mathcal{C}_1 = \overline{\mathcal{C}}_0$ for the first two cases

7.2 Computer experiments. Limited computer experiments have been performed to investigate realizable lengths of loops for small n. For each $5 \le n \le 8$, clutters were randomly generated and their loops were computed. The number of clutters generated are different for each n: more than 200 loops for $n = 5$, 6, less than 100 for $n = 7$, and one for $n = 8$. Table 2 contains this result. It seems that for $n = 5$, 2, 7 and 16 are the only realizable lengths. As n increases, the maximal length of loops grows rapidly. Also the number of different realizable lengths seems to grow fast. For $n = 6$, all the generated lengths are even except for one loop of length 87.

TABLE 2. Lengths of known examples of loops

n	Lengths
1	<u>3</u>
2	2, <u>4</u>
3	<u>5</u>
4	2, 3, <u>6</u>
5	2, <u>7</u>, 16
6	2, 4, 6, <u>8</u>, 10, 12, 14, 16, 18, 24, 28, 32, 34, 40, 46, 48, 52, 54, 56, 60, 64, 66, 68, 70, 76, 78, 82, 87, 90, 92, 94, 102, 104, 124, 128, 132, 134, 252, 380, 414, 616, 1026, 1032
7	5, 8, <u>9</u>, 22, 27, 36, 43, 45, 46, 65, 124, 144, 170, 277, 396, 444, 923, 1083, 1229, 1331, 2685, 3025, 3791
8	2, 5, <u>10</u>, 186
9	2, 5, <u>11</u>

Note: The table contains all realizable lengths for $n \le 4$. The underlined length $n+2$ is always realizable.

Table 3 contains some realizable lengths up to $(n+2)$ for $n \le 9$. The lengths 3 and 4 seem to be quite exceptional, because excluding the complete uniform loops, there is only one loop for each length.

TABLE 3. Lengths (up to $n+2$) of known examples of loops of k-uniform clutters

```
n  k   Lengths                  Attained by
---------------------------------------------------------------
1  1     3
2  1   2  4
       L-------------------- Duchet (1)
3  1         5
4  1           6
   2   2 3     6
       | L------------------ closed snake (12,23,31)
       L-------------------- Duchet (13,14,23,24) and
                             open snake (12,23,34)
5  1              7
   2   2          7
       L-------------------- closed snake (12,23,34,45,51)
6  1                 8
   2                 8
   3   2   4   6     8
       |   L---------------- Star(1,3,6)
       L-------------------- Duchet (123,145,146,156,
                                     245,246,256,
                                     345,346,356)
7  1                    9
   2                 8  9
   3         5       8  9
             |       L------ closed snake (12,23,34,45,56,67,71)
             L-------------- Fano=S(2,3,7)
8  1                      10
   2                      10
   3                      10
   4   2     5            10
       |     L-------------- Star(1,4,8)
       L-------------------- Duchet
9  1                        11
   2                        11
   3   2     5              11
       |     L-------------- S(2,3,9)
       L-------------------- Ex.4.5 in (Du).
```

Table 4 shows average sizes of loops $\|L\| \;/\; |L|$. It is remarkable that average sizes are quite close to each other. We observed that the complete uniform loop attains the minimum. We conjecture that this is true in general.

TABLE 4. Average size of clutters ($\|L\|/|L|$)

n	$2^n/(n+2)$ (lower bound?)	Experiments Min Aver Max	$\binom{n}{\lfloor n/2 \rfloor}/2$ (Upper bound?)
2	1.00	$(1.00 = 1.00 = 1.00)$	1.00
3	1.60	$(1.60 = 1.60 = 1.60)$	1.60
4	8/3	$(8/3 < A_4 < 3.00)$	3.00 (Duchet)
5	4.57	$4.57 < 4.67 < 5.00$	5.00 (closed snake)
6	8.00	$8.00 < 8.26 < 10.00$	10.00 (Duchet)
7	14.22	$14.22 < 14.69 < 15.77$	17.50 (Fano gives only 16.80)
8	25.60		35.00 (Duchet)
9	46.55		63.00 (Ex. 4.5 (Du) gives 42.00, S(2, 3, 9) gives 55.20)

Note: The experiments were done for $n = 5, 6, 7$, and the values in brackets for $n = 2, 3, 4$ are exact. The average value A_4, for $n = 4$, is

$$(3a + (8/3)b)/(a + b),$$

where a is the number of loops of length 2 (types L4–1, L4–2 from Table 1) and b is the number of all other loops (all remaining 5 types).

8. Final Remarks. We have described many interesting problems on loops of clutters. Among them, the reflection conjecture is certainly the most challenging problem. In fact, conjecture (1.1)(a) can be strengthened. That is, for any loop L containing a uniform clutter:

(8.1) The reflection loop $\widetilde{L}$ exists, and furthermore for $\mathcal{C} \in L$ and $\mathcal{D} \in \widetilde{L}$, there is an integer M such that the correspondence between the k-uniform clutters of L and the k-uniform clutters of $\widetilde{L}$ is explicitly written as:

$$\widetilde{\mathcal{C}}_i = \mathcal{D}_{m-k-i},$$

for each k.

We verified this conjecture for most examples in Section 5.

In all known examples of p-self-dual-reflection $\widetilde{L} = \overline{L}^p$ (see definition in Section 5) we have, for some integer m, $\mathcal{C}_i = \widetilde{\widetilde{\mathcal{C}}}^p_{m+k+i}$ for any k-uniform $\mathcal{C}_i \in L$. Denote by $\pi(= |\langle p^2 \rangle|)$ the order of permutations p^2, i.e., the smallest integer with $(p^2)^\pi = \langle 1 \rangle$. Then, in such case, the length $t(L)$ of the loop divides $\pi(n+2m)$, because

$$\begin{aligned}
\mathcal{C}_i &= \widetilde{\widetilde{\mathcal{C}}}^p_{m+k+i} = \overline{\widetilde{\widetilde{\mathcal{C}}}^p_{m+(n-k)+(m+k+i)}}^{\,p} \\
&= \mathcal{C}^{p^2}_{(n+2m)+i} = \mathcal{C}^{(p^2)^\pi}_{\pi(n+2m)+i} \\
&= \mathcal{C}^{\langle 1 \rangle}_{\pi(n+2m)+i}
\end{aligned}$$

It is an example of what is called in [D–F2] a partial (on k-uniform clutters only) rotation.

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