ON VOLUME-MEASURE AS HEMI-METRICS *

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Abstract

Let m be a positive integer, and X be a point-set in a Euclidean space containing at least $m+2$ points. Let $\mu_m : X^{m+1} \to [0, \infty)$ be the map defined by $\mu_m(x_1, \ldots, x_{m+1})$ to be the m-dimensional volume of the convex hull of the point-set $\{x_1, \ldots, x_{m+1}\}$. Denote by $s_m = s_m(X)$ the maximum value of s such that

$$s \times \mu_m(x_1, \ldots, x_{m+1}) \le \sum_{i=1}^{m+1} \mu_m(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{m+2})$$

holds whenever $x_1, x_2, \ldots, x_{m+2}$ are mutually distinct. If μ_m is identically zero, then put $s_m(X) = \infty$. (This $s_m(X)$ is considered as a "bound" of the m-simplex inequality for μ_m on X.) We prove that if $s_2(X) = 3$ and $|X| \ge 5$, then X is the vertex set of a regular simplex, and present the values of s_m of the vertex-sets of several convex polytopes. For the vertex-set of the n-dimensional octahedron, we have $s_m = 3$ for all $n \ge m \ge 3$, while for the vertex set of the n-dimensional cube, we show $s_m \to 1$ as $n \to \infty$ for every $m > 0$.

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1 Introduction

Let m be a positive integer, and X be a finite point-set in a Euclidean space with size $|X| \geq m+2$. A map

$$\rho : X^{m+1} \to [0, \infty)$$

is called an *m-hemi-metric* if (i) ρ is totally symmetric, that is,

$$\rho(x_1, x_2, \ldots, x_{m+1}) = \rho(x_{\pi(1)}, \ldots, x_{\pi(m+1)})$$

for any permutation π of $\{1, 2, \ldots, m+1\}$, and (ii) ρ satisfies the m-simplex inequality, that is,

$$\rho(x_1, \ldots, x_{m+1}) \leq \sum_{i=1}^{m+1} \rho(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{m+2})$$

holds for any $x_1, \ldots, x_{m+2} \in X$. In the following we further impose the condition that if $x_1, \ldots, x_{m+1}$ are not mutually distinct, then $\rho(x_1, \ldots, x_{m+1}) = 0$. An m-hemi-metric ρ is called an *(m, s)-super-metric* on X if

$$s \times \rho(x_1, \ldots, x_{m+1}) \leq \sum_{i=1}^{m+1} \rho(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{m+2})$$

holds whenever $x_1, \ldots, x_{m+2}$ are all distinct. Thus an m-hemi-metric ρ is always an $(m, 1)$-super-metric by the m-simplex inequality. If ρ is not identically zero, the maximum value of s such that ρ is an (m, s)-super-metric on X is called the *super-bound* of ρ and denoted by $s(\rho, X)$. If ρ is identically zero, then we put $s(\rho, X) = \infty$.

Theorem 1 *If ρ is an m-hemi-metric on X that is not identically zero, then $s(\rho, X) \leq m+1$.*

The notion of m-hemi-metric was introduced in [2] as a small modification of well known notion of m-metric (see e.g. [4],[5],[6]), and the notion of (m, s)-super-metric was introduced in [1].

An example of an m-hemi-metric, which is the one we are going to consider in the following, is the m-dimensional Euclidean volume μ_m, that is, $\mu_m(x_1, \ldots, x_{m+1})$ is the m-dimensional volume of the convex hull of the point-set $\{x_1, \ldots, x_{m+1}\}$. Thus μ_1 is the usual Euclidean distance.

Let us denote the super-bound $s(\mu_m, X)$ simply by $s_m(X)$. The following two theorems show that in some cases, the value of $s_m(X)$ determines the 'configuration' X to some extent.

Theorem 2 *Let A be a set of $m+2$ points. Then $s_m(A) = 1$ if and only if the convex hull of A is an m-dimensional simplex.*

Theorem 3 *For a point set X containing at least 5 points, the following three conditions are equivalent.*

(0) *X is the vertex set of a regular simplex.*

(1) $s_1(X) = 2$.

(2) $s_2(X) = 3$.

Remark. (i) The assumption $|X| \geq 5$ is necessary, because the condition (2) also holds for the 4 vertices of a parallelogram. (ii) For any $k \geq 5$, there is an X of size k that is *not* the vertex set of a regular simplex, but $s_3(X) = 4$.

Problem. For $m > 3$, does the condition

(m) $\quad s_m(X) = m + 1$

imply the condition (0) of Theorem 3?

It will be of some interest to find $s_m(X)$ for several typical point-sets X. Using computer, we found the values of s_m for the vertex set of five Platonic solids:

vertex-set of	s_1	s_2	s_3
tetrahedron	2	3	-
octahedron	$\sqrt{2}$	$1+\sqrt{3}$	3
cube	$\frac{1+\sqrt{2}}{\sqrt{3}}$	$\sqrt{3}$	2
dodecahedron	$\sqrt{2}\frac{\sqrt{5}+2}{\sqrt{5}+3}$	$\sqrt{3}\frac{\sqrt{22\sqrt{5}+50}}{3\sqrt{5}+7}$	$\frac{27\sqrt{5}+61}{21\sqrt{5}+47}$
icosahedron	$\frac{4}{1+\sqrt{5}}$	$\frac{\sqrt{6\sqrt{5}+30+2\sqrt{24\sqrt{5}+60}}}{3\sqrt{5}+9}$	$\frac{2\sqrt{5}+3}{\sqrt{5}+2}$.

For the vertex set V of the 4-dimensional polytope 24-cell, we have

$$s_1(V) = \frac{2}{\sqrt{3}},\ s_2(V) = \frac{1+2\sqrt{2}}{\sqrt{5}},\ s_3(V) = \frac{1+3\sqrt{2}}{\sqrt{7}},\ s_4(V) = \frac{5}{3}.$$

Let O_n be the vertex set of the n-dimensional octahedron and let Q_n denote the vertex set of the n-dimensional cube in R^n.

Theorem 4 *For $n > m$,*

$$\begin{aligned} s_m(O_n) &= \begin{cases} \sqrt{2} & m = 1 \\ 1+\sqrt{3} & m = 2 \\ 3 & m \geq 3 \end{cases} \\ s_m(Q_n) &\leq \begin{cases} \sqrt{3} & m = 2 \\ 2 & m \geq 3. \end{cases} \end{aligned}$$

Remark. For the cube-case, we have the following exact values:
$s_2(Q_4) = \frac{1+2\sqrt{2}}{\sqrt{5}}, s_2(Q_5) = \frac{1+2\sqrt{3}}{\sqrt{7}}, s_2(Q_6) = \frac{1+2\sqrt{4}}{\sqrt{9}}$,
$s_2(Q_7) = \frac{1+2\sqrt{5}}{\sqrt{11}}, s_2(Q_8) = \frac{1+2\sqrt{6}}{\sqrt{13}}$,
$s_3(Q_4) = \frac{1+3\sqrt{2}}{\sqrt{7}}, s_3(Q_5) = \frac{1+3\sqrt{3}}{\sqrt{10}}, s_3(Q_6) = \frac{1+3\sqrt{4}}{\sqrt{13}}, s_3(Q_7) = \frac{1+3\sqrt{5}}{\sqrt{16}}$,
$s_4(Q_5) = \frac{1+4\sqrt{2}}{\sqrt{13}}, s_4(Q_6) = \frac{\sqrt{2}+\sqrt{3}+3\sqrt{4}}{\sqrt{25}}$.

The exact values for s_2, s_3 seem to suggest the formulas

$$\frac{1+2\sqrt{n-2}}{\sqrt{2n-3}}, \quad \frac{1+3\sqrt{n-2}}{\sqrt{3n-5}},$$

for $s_2(Q_n), s_3(Q_n)$, respectively. (In fact, by following the patterns of the vertices giving these values, we can prove that they are upper bounds of $s_2(Q_n)$ and $s_3(Q_n)$.) But they do not give the correct values for large n since we have the following asymptotic result.

Theorem 5 *For any fixed $m > 0$, $\lim_{n\to\infty} s_m(Q_n) = 1$.*

2 Proofs of Theorems 1, 2 and 3

For $B = \{x_1, \ldots, x_{m+1}\}$, we may write $\rho(B)$ instead $\rho(x_1, \ldots, x_{m+1})$. Notice that if $s(\rho, X) < \infty$, then $s(\rho, X)$ is the minimum value of $s(\rho, A)$ for $(m+2)$-set $A \subset X$, where

$$s(\rho, A) = \min_{y\in A} \left\{ \frac{\sum_{x\in A-\{y\}} \rho(A - \{x\})}{\rho(A-\{y\})} \right\}.$$

Proof of Theorem 1. Since ρ is not identically zero, there is an $(m+2)$-set $A \subset X$ such that ρ is not identically zero on A. Let $B_1, B_2, \ldots, B_{m+2}$ be the $(m+1)$-subsets of A. We may suppose that

$$\rho(B_1) \leq \rho(B_2) \leq \ldots \leq \rho(B_{m+2}).$$

Then $\rho(B_{m+2}) > 0$, and

$$s(\rho, A) \leq \frac{\rho(B_1) + \ldots + \rho(B_{m+1})}{\rho(B_{m+2})} \leq m + 1,$$

which proves Theorem 1.

Proof of Theorem 2. Denote the convex hull of X by $conv(X)$.

Suppose $s_m(A) = 1$. Then $\dim(conv(A)) \geq m$, for otherwise, we should have $s_m(A) = \infty$ by definition. If $\dim(conv(A)) \geq m + 1$, then $conv(A)$ is an $(m + 1)$-dimensional simplex. In this case, it is easy to see that $s_m(A) > 1$, a contradiction. Therefore we can deduce that $\dim(conv(A)) = m$. If $conv(A)$ is not an m-dimensional simplex, $m + 2$ points are all vertices of the polytope $conv(A)$. Then, it is not difficult to see that for any $y \in A$, $conv(A - \{y\})$ is a proper subset of $\bigcup_{x \in A - \{y\}} conv(A - \{x\})$. Hence we also have $s_m(A) > 1$, a contradiction. Thus, $conv(A)$ must be an m-dimensional simplex.

Conversely, if $conv(A)$ is an m-dimensional simplex, then there is a $y \in A$ such that $y \in conv(A - \{y\})$. Then $conv(A - \{y\}) = \bigcup_{x \in A - \{y\}} conv(A - \{x\})$. Hence we have $s_m(A) = 1$.

Proof of Theorem 3. Since (0) $\Leftrightarrow$ (1) and (0) $\Rightarrow$ (2) are obvious, we show (2) $\Rightarrow$ (0). Let x, y, z, w be four distinct points in X, and suppose that among the triples in $\{x, y, z, w\}$, μ_2 becomes maximum at $\{x, y, z\}$. Then the inequality

$$3 \times \mu_2(x, y, z) \leq \mu_2(x, y, w) + \mu_2(x, z, w) + \mu_2(y, z, w)$$

implies that μ_2 is constant on the triples in $\{x, y, z, w\}$. Thus, for any four distinct points x, y, z, w, we have $\mu_2(x, y, z) = \mu_2(x, y, w)$. Therefore, for any two triples $\{x, y, z\}$ and $\{u, v, w\}$ in X, we have $\mu_2(x, y, z) = \mu_2(u, v, w)$. Since $s_2(X) < \infty$ implies that μ_2 is not identically zero, the value of μ_2 is a positive constant.

Now Theorem 3 follows from the result obtained in [3]: *If* $|X| \geq 5$ *and all triangles, each spanned by a triple of* X*, have the same area* > 0*, then* X *forms the vertex set of a regular simplex.*

As given in an example in [3], the set

$$X = \{(\pm 1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), (0, 0, 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)\}$$

satisfies $s_3(X) = 4$. Since we cannot think of any other example, it seems true that if $s_3(X) = 4$, then $X - \{\text{one point}\}$ is the vertex set of a regular simplex.

3 Octahedra

Here we show the O_n case of Theorem 4. We may put

$$O_n = \{(\pm 1, 0, \ldots, 0), (0, \pm 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, \pm 1)\},$$

Let us find the minimum value of

$$\frac{\sum_{x \in A - \{y\}} \mu_m(A - \{x\})}{\mu_m(A - \{y\})}$$

for $(m+2)$-set $A \subset O_n$ and $y \in A$. Since $s_1(O_n) = \sqrt{2}$ is obvious, we assume that $m \geq 2$. Denote the vertices of O_n by p_i, q_i, $i = 1, 2, \ldots, n$, where p_i has i-th coordinate $+1$ and other coordinates 0, and q_i has i-th coordinate -1. Notice that we do not need to consider the case, in which the dimension of the convex hull of A is less than m. If the dimension of the convex hull of A is $\geq m$, then A cannot contain three pairs of vertices q_i, p_i; q_j, p_j; q_k, p_k. So, we may consider only such an A that is *congruent* to one of the following three sets A_1, A_2, A_3:

$$\begin{aligned}
A_1 &= \{p_1, p_2, p_3, p_4, \ldots, p_{m+2}\} \text{ (all suffixes are different)}, \\
A_2 &= \{q_1, p_1, p_2, p_3, \ldots, p_{m+1}\} \text{ (only one suffix appears twice)}, \\
A_3 &= \{q_1, p_1, q_2, p_2, p_3, \ldots, p_m\} \text{ (only two suffixes appear twice)}.
\end{aligned}$$

Since A_1 spans an $(m+1)$-dimensional regular simplex,

$$s_m(A_1) = m + 1.$$

The third set A_3 contains 4 vertices q_1, p_1, q_2, p_2 forming a square, and A_3 spans an m-dimensional polytope. If an $(m+1)$-subset B of A_3 contains the 4 vertices q_1, p_1, q_2, p_2, then B spans $(m-1)$-dimensional polytope, and $\mu_m(B) = 0$. So, in order to get a B with nonzero μ_m value, we have to remove one of q_1, p_1, q_2, p_2 from A_3. Thus among the $(m+1)$-subsets of A_3, exactly four of them have positive μ_m values, and their values are all equal. Hence

$$s_m(A_3) = 3.$$

Now we consider the set A_2. Since $A_2 - \{q_1\}$ spans a regular m-simplex of edge-lenght $\sqrt{2}$, we have

$$\mu_m(A_2 - \{q_1\}) = \mu_m(A_2 - \{p_1\}) = \frac{\sqrt{m+1}}{m!},$$

and

$$\mu_m(A_2 - \{p_2\}) = \mu_m(A_2 - \{p_3\}) = \ldots = \mu_m(A_2 - \{p_{m+1}\}) = \frac{2}{m!}.$$

Hence, for $m = 2$,

$$s_2(A_2) = \frac{2/2! + 2\sqrt{3}/2!}{2/2!} = 1 + \sqrt{3},$$

and for $m \geq 3$,

$$s_m(A_2) = \frac{\sqrt{m+1}/m! + m(2/m!)}{\sqrt{m+1}/m!} = 1 + \frac{2m}{\sqrt{m+1}} \geq 4.$$

Therefore, $s_2(O_n) = 1 + \sqrt{3}$ and $s_m(O_n) = 3$ for $m \geq 3$.

4 Cubes

Let us consider the cube case of Theorem 4. We may suppose that Q_n is the vertex set of the n dimensional cube spanned by the n vectors

$$(1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)$$

in R^n. To get the first inequality, consider the set of 4 points

$$\begin{aligned} &(0, 0, 0, 0, \ldots, 0), \\ &(1, 0, 0, 0, \ldots, 0), \\ &(0, 1, 0, 0, \ldots, 0), \\ &(0, 0, 1, 0, \ldots, 0). \end{aligned}$$

The value of s_2 on this set is

$$\frac{1/2 + 1/2 + 1/2}{\sqrt{3}/2} = \sqrt{3}.$$

Hence $s_2(Q_n) \leq \sqrt{3}$.

Now suppose $m \geq 3$. To see clearly, let us consider the case $m = 5, n = 8$. Let A be the set of the following $m+2 = 7$ vertices of Q_8:

$$\begin{array}{rcl} p_0 & = & (0,0,0,0,0,0,0,0), \\ p_1 & = & (1,0,0,0,0,0,0,0), \\ p_2 & = & (0,1,0,0,0,0,0,0), \\ p_3 & = & (0,0,1,0,0,0,0,0), \\ p_4 & = & (1,1,1,0,0,0,0,0), \\ p_5 & = & (0,0,0,1,0,0,0,0), \\ p_6 & = & (0,0,0,0,1,0,0,0). \end{array}$$

To compute $s_5(A)$, we need the values $\sigma_i := \mu_5(A - \{p_i\}), i = 0, 1, \ldots, 6$. It is obvious that $\sigma_5 = \sigma_6 = 0$ from their dimensions. It is also easy to see that $\sigma_4 = 1/5!$. Now, since the 3-dimensional volume of the simplex spanned by p_1, p_2, p_3, p_4 is $1/3 = 2/3!$, we have $\sigma_0 = \frac{2}{3!} \times \frac{1}{4} \times \frac{1}{5} = \frac{2}{5!}$. Similarly, since the 3-dimensional volume of the simplex spanned by p_0, p_2, p_3, p_4 is $1/3!$, we have $\sigma_1 = \sigma_2 = \sigma_3 = 1/5!$. Therefore

$$s_5(A) = \frac{1/5! + 1/5! + 1/5! + 1/5!}{2/5!} = 2,$$

which proves $s_5(Q_8) \leq 2$.

Similar argument clearly works in any $n > m \geq 3$.

5 Cubes – asymptotic case

Here we show that for any fixed $m > 0$, $s_m(Q_n) \to 1$ as $n \to \infty$. To be clear, we put $m = 3, n = 3k+1$. Let

$$\begin{array}{rcl} p_0 & = & (0, \overbrace{0,\ldots,0}^{k}, \overbrace{0,\ldots,0}^{k}, \overbrace{0,\ldots,0}^{k}), \\ p_1 & = & (0,1,\ldots,1,0,\ldots,0,0,\ldots,0), \\ p_2 & = & (0,0,\ldots,0,1,\ldots,1,0,\ldots,0), \\ p_3 & = & (0,0,\ldots,0,0,\ldots,0,1,\ldots,1), \\ p_4 & = & (1,0,\ldots,0,0,\ldots,0,0,\ldots,0). \end{array}$$

Then, since $p_0p_i = \sqrt{k}$, $p_4p_i = \sqrt{k+1}$ for $i = 1, 2, 3$, and $p_0p_4 = 1$, it is easy to see that

$$\mu_m(p_0, p_1, p_2, p_3) = \frac{k^{m/2}}{m!}, \quad \mu_m(p_1, p_2, p_3, p_4) \ \leq \ \frac{(k+1)^{m/2}}{m!},$$

$$\begin{aligned} \mu_m(p_0, p_2, p_3, p_4) &= \mu_m(p_0, p_1, p_3, p_4) \\ &= \mu_m(p_0, p_1, p_2, p_4) = \frac{(k+1)^{(m-1)/2}}{m!}. \end{aligned}$$

Hence

$$1 \leq s_m(Q_{3k+1}) \leq \frac{(k+1)^{m/2} + m(k+1)^{(m-1)/2}}{k^{m/2}}.$$

Since the right-most-side tends to 1 as $k \to \infty$, $s_m(Q_n)$ tends to 1 as $n \to \infty$. The same argument works for any $m > 0$.

References

[1] M. Deza and M. Dutour, Cones of metrics, hemi-metrics and super-metrics, Annals of European Academy of Sciences (2003) 141-162.

[2] M. Deza and G. Rosenberg, n-semi-metrics, European J. Combinatorics, Special Issue "Discrete metric spaces" 21-6(2001), 797-806.

[3] P. Frankl and H. Maehara, Simplices with given 2-face areas, European J. Combinatorics 11(1990), 241-247.

[4] A. Froda, Espaces p-métriques et leur topologie, Comptes Rendus de l'Acad. Sci. Paris 247(1958) 849-852.

[5] S. Gähler, 2-metric spaces and their topological structure (in German), Math. Nachr., 26(1963) 115-148.

[6] K. Menger, Untersuchungen über allegemeine Metrik, Math. Ann., 100(1928) 75-165.

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