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The inequicut cone*

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Abstract

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Given a complete graph K_n on n nodes and a subset S of nodes, the cut $\delta(S)$ defined by S is the set of edges of K_n with exactly one endnode in S. A cut $\delta(S)$ is an equicut if $|S| = \lfloor n/2 \rfloor$ or $\lceil n/2 \rceil$ and an inequicut, otherwise. The cut cone C_n (or inequicut cone IC_n) is the cone generated by the incidence vectors of all cuts (or inequicuts) of K_n. The equicut polytope EP_n, studied by Conforti et al. (1990), is the convex hull of the incidence vectors of all equicuts. We prove that IC_n and EP_n 'inherit' all facets of the cut cone C_n, namely, that every facet of the cut cone C_n yields (by zero-lifting) a facet of the inequicut cone IC_m for $n < \lfloor m/2 \rfloor$ and of EP_m for m odd, $m \geqslant 2n+1$. We construct several new classes of facets, not arising from C_n, for the inequicut cone IC_n and we describe its facial structure for $n \leqslant 7$.

0. Introduction

Given the complete graph K_n on n nodes $1, 2, \ldots, n$, and a subset S of nodes, the cut $\delta(S)$ defined by S is the set of edges of K_n with exactly one endnode in S and the cut vector defined by S is its incidence vector. The cut cone C_n (or the cut polytope P_n) is the cone (or the polytope) generated by *all* cut vectors of K_n; both objects were considered by many authors (see for instance [4, 5, 8, 9, 11, 12]). In this paper, we consider some *restricted* cut cones or polytopes, i.e. cones and polytopes generated by

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some family of cuts. The most natural families of cuts are those defined by imposing some restriction on the size $|S|, n-|S|$ of the shores $S, \bar{S}$. For example, equicuts (or inequicuts), k-uniform cuts, even cuts (or odd cuts), are the cuts $\delta(S)$ for which $|S|=\lfloor n/2 \rfloor$ or $\lceil n/2 \rceil$, (or $|S|\neq\lfloor n/2 \rfloor, \lceil n/2 \rceil$), $|S|=k$ or $n-k$, both $|S|$, $n-|S|$ are even (or odd), respectively.

In this paper, we study the facial structure of the inequicut cone IC_n (i.e. the cone generated by the inequicuts of K_n) and, in continuation of [7], of the equicut polytope EP_n (i.e. the convex hull of the equicuts of K_n). The k-uniform cut cone C_n^k as well as the lattice generated by the k-uniform cuts are studied in details in [14]; we describe here the cone C_n^k for $k=1, 2$, they are actually the only simplicial k-uniform cut cones together with the case ($n=6$, $k=3$). The cone and polytope generated by even or odd cuts are considered in [13].

One of the reasons for which equicuts (or inequicuts) are useful comes from their application to the study of the ground state magnetisation of a spin glass with zero (or non zero) magnetisation (for a survey of applications, see [3]). Another interesting feature of the inequicut cone and of the equicut polytope is that they already contain, in some sense, all the facets of the cut cone in smaller dimension.

The paper is organized as follows. In Section 1, we show that the inequicut cone and the equicut polytope 'inherit' all the facets of the cut cone; namely, we prove that any facet of the cut cone C_n yields (by zero-lifting) a facet of the inequicut cone IC_m for $n<\lfloor m/2 \rfloor$ and of the equicut polytope EP_m for m odd, $m\geqslant 2n+1$. In Section 2; we give the full description of the k-uniform cut cone for $k=1, 2$, from which we can deduce the facial structure of the inequicut cone IC_n and of the equicut polytope EP_n for $n\leqslant 5$. Several classes of facets of the inequicut cone IC_n are described in Section 3. Actually, they were obtained by inspection and generalization of some facets for IC_6, IC_7; so, we also give the full description of IC_n for $n=6$, 7. Moreover, for each class of facets, we consider its equality case, i.e. we describe the inequicuts satisfying the inequality at equality (the roots). A nice feature of the new classes of facets is that they can all be triangulated, i.e. decomposed as linear combinations of triangles (see Section 3.6); in fact, this property is very useful for proving validity and identifying the roots. Section 4 contains the proofs for the classes of facets presented in Section 3.

We now give some basic notation and preliminaries needed for the paper; for more details on the cut cone and polytope, see [11] and on the equicut polytope, see [7]. Given a subset S of $[1, n]$, the cut vector $\chi^{\delta(S)}$ defined by S is the incidence vector of the cut $\delta(S)$, i.e. the 0–1 vector of length $\binom{n}{2}$ defined by $\chi_{ij}^{\delta(S)}=1$ if $i\in S, j\notin S$ or $i\notin S, j\in S$ and $\chi_{ij}^{\delta(S)}=0$ otherwise, for $1\leqslant i<j\leqslant n$. The cut (or inequicut, equicut) cone is the cone C_n (resp. IC_n, EC_n) generated by all cut (or inequicut, equicut) vectors; similarly, the cut (or inequicut, equicut) polytope is the polytope P_n (or IP_n, EP_n) which is the convex hull of all cut (or inequicut, equicut) vectors. Observe that C_n, IC_n, EC_n are cones pointed at the origin and that the polytopes P_n, IP_n contain the origin (since the inequicut $\delta([1, n])$ is the zero vector) while the equicut polytope EP_n does not contain the origin. So, the facets of IC_n are precisely the facets of IP_n containing the origin and thus the study of IC_n is also a contribution to the study of IP_n. Given disjoint subsets

S, T of $[1, n]$, we denote by $\delta(S, T)$ the set of edges (i, j) with $i \in S$, $j \in T$; so, $\delta(S, N-S) = \delta(S)$. Given a vector v in $R^{\binom{n}{2}}$ and a set E of edges of K_n, we set $v(E) = \sum_{(i,j) \in E} v_{ij}$.

Given a subset X of $R^{\binom{n}{2}}$, its dimension $\dim(X)$ is the maximum number of affinely independent vectors in X minus one and, if X contains the origin, then $\dim(X)$ is also the maximum number of linearly independent vectors in X; if $\dim(X) = n(n-1)/2$, X is called full dimensional. Given a vector v in $R^{\binom{n}{2}}$, a scalar α and a subset X of $R^{\binom{n}{2}}$, the inequality $v.x \leqslant \alpha$ is called valid over X if it is satisfied by all vectors of X, i.e. by all cut (or inequicut, equicut) vectors if $X = C_n$ or P_n (or IC_n or IP_n, EC_n or EP_n), the cuts (resp. inequicuts, equicuts) whose incidence vectors satisfy the equality $v.x = \alpha$ being then called the roots of the inequality $v.x \leqslant \alpha$. Note that, if X is a cone (for instance, C_n, IC_n or EC_n), then X is defined by homogeneous inequalities, i.e. of the form $v.x \leqslant 0$. If X is a cone or a polytope and $v.x \leqslant \alpha$ is a valid inequality over X, the set $F_v = \{x \in X: v.x = \alpha\}$ is the face of X induced by the inequality $v.x \leqslant \alpha$ and F_v is a facet if $\dim(F_v) = \dim(X) - 1$. The polyhedra C_n, P_n, IC_n, IP_n are full dimensional while EP_n is not full dimensional. In fact, EP_{2p} has dimension $\binom{2p}{2} - 2p$, while EP_{2p+1} has dimension $\binom{2p+1}{2} - 1$; furthermore, EP_{2p+1} is the facet of P_{2p+1} induced by the inequality: $\sum_{1 \leqslant i < j \leqslant 2p+1} x_{ij} \leqslant p(p+1)$ ([7]). It is also mentioned in [7] that the facial description of EP_{2p} can be deduced from that of EP_{2p-1} and, henceforth, it is enough to study facets of EP_n for n odd. Finally, a cone is said to be simplicial if its generators are linearly independent and a simplicial polytope has all its vertices affinely independent; hence, a simplicial facet of a cone or a polytope has all its roots affinely independent.

1. Some relations between facets of P_n, IC_n and EP_n

In this section, we study how all the facets of the cut cone can be transported to the inequicut cone (or polytope) or the equicut polytope. Given $v \in R^{\binom{n}{2}}$, any facet defining inequality $v.x \leqslant 0$ for the cone C_n defines trivially a valid inequality for the inequicut cone IC_n (or polytope IP_n) and the equicut polytope EP_n. We first give a result stating that every facet of C_n, in fact, defines a facet of the inequicut cone IC_m, provided m is large enough, or more explicitly, $\lfloor m/2 \rfloor > n$. Conversely, if the inequality $v.x \leqslant 0$ is facet defining for the inequicut IC_n, then it is facet defining for the cut cone C_n if and only if it is valid for C_n. We will see in Section 4 several examples of facets of IC_n which, however, are not valid for C_n, i.e. which are violated by some equicuts. Given integers $m > n \geqslant 2$ and $v \in R^{\binom{n}{2}}$, we set $v' = (v, 0, \ldots, 0) \in R^{\binom{m}{2}}$, i.e. $v'_{ij} = v_{ij}$ for $1 \leqslant i < j \leqslant n$ and $v'_{ij} = 0$ for $i < j, 1 \leqslant i \leqslant m$ and $n+1 \leqslant j \leqslant m$.

Theorem 1.1. *Given $v \in R^{\binom{n}{2}}$, assume that the inequality $v.x \leqslant 0$ defines a facet of the cut cone C_n. Then, the inequality $v'.x \leqslant 0$ defines a facet of the inequicut cone IC_m for all $n < \lfloor m/2 \rfloor$.*

Proof. Set $l=m-n$, $N=[1,n]$, $L=[1',l']$, consisting of the l elements $1',2',\ldots,l'$, and $M=N\cup L$; so C_n is the cut cone defined on the n nodes of N and IC_m is the inequicut cone defined on the m nodes of M. In order to prove that the inequality $v'.x\leqslant 0$ defines a facet of IC_m, we shall exhibit a set R of

$$\binom{m}{2}-1=\binom{n}{2}-1+(n-1)l+\binom{l+1}{2}$$

linearly independent roots of $v'.x\leqslant 0$.

First, since $v.x\leqslant 0$ defines a facet of C_n, we can find a set R'_1 of $\binom{n}{2}-1$ roots $\delta(S_i)$, $1\leqslant i\leqslant\binom{n}{2}-1$, whose incidence vectors are linearly independent, where S_i is a proper subset of N. Let A denote the $(\binom{n}{2}-1)\times\binom{n}{2}$ incidence matrix of the cut vectors $\chi^{\delta(S_i)}$, $1\leqslant i\leqslant\binom{n}{2}-1$; its rank is $\binom{n}{2}-1$, hence all its column except one are linearly independent. Therefore, we can suppose that the $n-1$ columns indexed by $I=\{(1,2);(1,3);\ldots;(1,n)\}$ and linearly independent; call A_1 the submatrix of A with column index I. We can assume w.l.o.g. that $1\notin S_i$ for all $1\leqslant i\leqslant\binom{n}{2}-1$ (replacing, if necessary, the set S_i by its complement $N-S_i$).

Since A_1 has full rank $n-1$, $n-1$ of its rows are linearly independent; for instance, the projections denoted by $\delta_1,\ldots,\delta_{n-1}$, of the incidence vectors of the cuts $\delta(S_1),\ldots,\delta(S_{n-1})$ on I are linearly independent. We denote by B this $(n-1)\times(n-1)$ nonsingular submatrix of A_1.

For all $1\leqslant i\leqslant\binom{n}{2}-1$, the cuts $\delta(S_i,M-S_i)$ of C_m are, in fact, inequicuts, since $|S_i|<n<\lfloor m/2\rfloor$; this forms a first set R_1 of $\binom{n}{2}-1$ roots of $v'.x\leqslant 0$.

Next, for each of the above sets $S_1,\ldots,S_{n-1}$, the cuts $\delta(S_i\cup\{h\})$, for $h\in L$, are inequicuts in IC_m which are roots of $v'.x\leqslant 0$; this gives a set R_2 of $(n-1)l$ roots.

Consider then the set R_3 consisting of the l cuts $\delta(h)$, for $h\in L$; they are obviously inequicuts and roots of $v'.x\leqslant 0$ in IC_m.

Finally, define the set R_4 consisting of the $\binom{l}{2}$ cuts $\delta(\{h,k\})$, for $h<k$, $h,k\in L$; they are obviously inequicuts and roots of $v'.x\leqslant 0$ in IC_m.

We now verify that the $\binom{m}{2}-1$ cut vectors associated with $R_1\cup R_2\cup R_3\cup R_4$ are linearly independent.

Let A' denote the $(\binom{m}{2}-1)\times\binom{m}{2}$ matrix whose rows are the incidence vectors of the cuts of $R_1\cup R_2\cup R_3\cup R_4$ (in that order); its columns being indexed by $J=\{(i,j)\colon i<j,\ i,j\in N\}$ (containing I in first $n-1$ positions), $H=H_1\cup\cdots\cup H_l$, where $H_h=\{(h,i)\colon i\in N\}$ for all $h\in L$ and $K=\{(h,k)\colon h\leqslant k,\ h,k\in L\}$. The matrix A' has the form shown in Fig. 1 (a matrix of all 1's is denoted by 1).

It is easy to see that the matrix C_i (see Fig. 1), for $1\leqslant i\leqslant n-1$, has the following block configuration:

$$C_i=\begin{bmatrix} 1-(0,\delta_i) & (0,\delta_i) & & (0,\delta_i) \\ (0,\delta_i) & 1-(0,\delta_i) & & (0,\delta_i) \\ \vdots & \vdots & \vdots & \vdots \\ (0,\delta_i) & (0,\delta_i) & & 1-(0,\delta_i) \end{bmatrix}$$

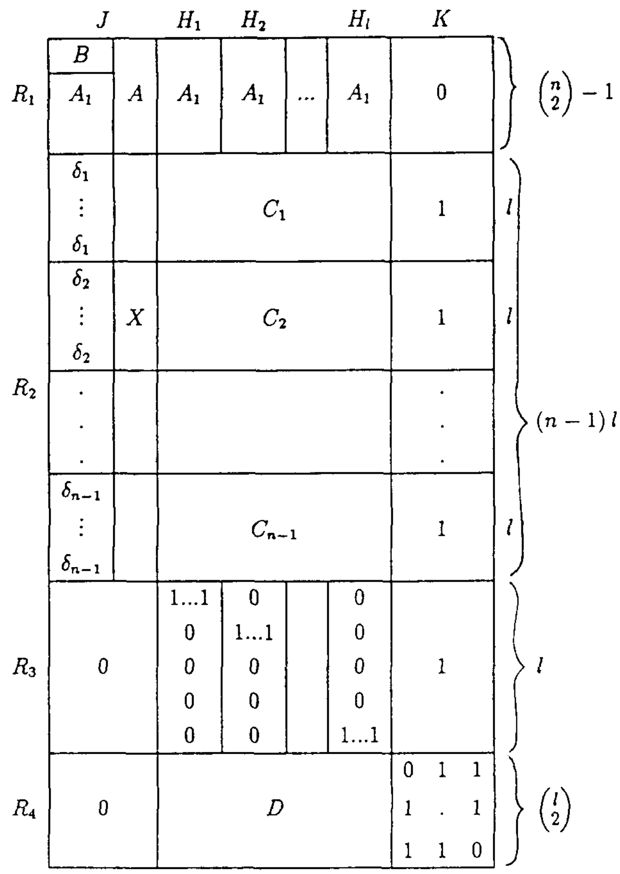

Fig. 1.

Hence, doing some column and row manipulation on A' brings it into the form shown in Fig. 2, where it is evident that A' has full rank $\binom{m}{2}-1$ (after rearranging rows of R_2, R_3). $\square$

The bound $n<\lfloor m/2 \rfloor$ of Theorem 1.1 can probably be improved, but not more than to $\lceil n/2 \rceil < \lfloor m/2 \rfloor$. Actually, any facet of C_6 is a facet of IC_8, but some facets of C_5 are not facets of IC_7.

We now turn to the connections between facets of the cut polytope P_n and the equicut polytope EP_n for $n=2p+1$, $p\geqslant 2$ integer. Since EP_{2p+1} is the facet of P_{2p+1} defined by $EP_{2p+1}=P_{2p+1}\cap\{x\colon \sum_{1\leqslant i<j\leqslant 2p+1} x_{ij}=p(p+1)\}$, it follows that every facet of EP_{2p+1} arises, in fact, as the intersection of some facet of P_{2p+1} with EP_{2p+1}. In other words, every facet F of EP_{2p+1} admits a unique (up to multiplication by a scalar) equation $v.x\leqslant\alpha$, i.e. $F=\{x\in EP_{2p+1}\colon v.x=\alpha\}$, such that the inequality $v.x\leqslant\alpha$ is valid for P_{2p+1} and induces a facet of P_{2p+1}; of course, every inequality

<table>
<tr><td></td><td>J</td><td>H_1</td><td>H_2</td><td></td><td>H_l</td><td>K</td></tr>
<tr><td>R_1</td><td>A</td><td colspan="4">0</td><td>0</td></tr>
<tr><td></td><td></td><td>$(0,-2\delta_1)$
0
.
0</td><td>0
$(0,-2\delta_1)$
.
0</td><td>.</td><td>0
0
.
$(0,-2\delta_1)$</td><td>0</td></tr>
<tr><td>R_2</td><td></td><td>$(0,-2\delta_2)$
0
.
0</td><td>0
$(0,-2\delta_2)$
.
0</td><td>.</td><td>0
0
.
$(0,-2\delta_2)$</td><td>0</td></tr>
<tr><td></td><td></td><td>$\vdots$</td><td>$\vdots$</td><td>$\vdots$</td><td>$\vdots$</td><td></td></tr>
<tr><td></td><td></td><td>$(0,-2\delta_{n-1})$
0
.
0</td><td>0
$(0,-2\delta_{n-1})$
.
0</td><td>.</td><td>0
0
.
$(0,-2\delta_{n-1})$</td><td>0</td></tr>
<tr><td>R_3</td><td>0</td><td>1...1
0
0
0
0</td><td>0
1...1
0
0
0</td><td>.</td><td>0
0
0
0
1...1</td><td>0</td></tr>
<tr><td>R_4</td><td>0</td><td colspan="4"></td><td>1 0 0
0 . 0
0 0 1</td></tr>
</table>

Fig. 2.

$Av.x+B\sum_{1\leqslant i<j\leqslant 2p+1}x_{ij}\leqslant A\alpha+Bp(p+1)$ with $A>0$, B scalars, induces the same facet F of P_{2p+1}, but does not define a facet of P_{2p+1} if $B\neq 0$.

For instance, all the known facets of EP_{2p+1} arise from the following classes of facets of the cut polytope P_{2p+1}:

— some hypermetric facets (in fact, some specific switching of them) (cf. [7, Theorems 7.2, 7.3, 7.7]),

— pure clique-web facets (in fact, some specific switching of them) (cf. [7, Theorems 6.1, 6.2, 6.3, 6.4] for the special classes of hypermetric, cycle and bicycle odd wheel inequalities and [12, Theorem 2.5] for the general case). (See e.g. [12] for a precise definition of these facets.)

Let us set $Q(b_1,\ldots,b_n).x=\sum_{1\leqslant i<j\leqslant n}b_ib_jx_{ij}$. For example, up to permutation, all the facets of EP_7 are:

(1) $Q(1,1,-1,0,0,0,0).x\leqslant 0$ (triangle inequality),

(2) $Q(1,1,1,0,0,0,0).x\leqslant 2$,

(3) $Q(1,1,1,1,1,0,0).x\leqslant 6$,

(4) $Q(2,1,1,1,1,1,0).x \leqslant 8$,
(5) $Q(2,2,1,1,1,1,1).x \leqslant 20$,
(6) $Q(1,1,1,1,1,1,1).x-(x_{12}+x_{23}+x_{34}+x_{45}+x_{51}) \leqslant 10$,
(7) $Q(2,2,1,1,1,1,1).x-(x_{12}+x_{23}+x_{34}+x_{41}) \leqslant 12$.

Facets(2)–(5) are switchings of the hypermetric inequalities $Q.x(1,1,-1,0,0,0,0) \leqslant 0$, $Q(1,1,1,-1,-1,0,0).x \leqslant 0$, $Q(2,1,1,-1,-1,-1,0).x \leqslant 0$, $Q(2,2,1,-1,-1,-1,-1).x \leqslant 0$, respectively. Facets (6) and (7) are switchings of the cycle inequalities $Q(1,1,1,1,1,-1,-1).x-(x_{12}+x_{23}+x_{34}+x_{45}+x_{51}) \leqslant 0$ and $Q(2,2,1,1,-1,-1,-1).x-(x_{12}+x_{23}+x_{34}+x_{41}) \leqslant 0$.

We now give the analogue of Theorem 1.1 for the equicut polytope.

Theorem 1.2. *Given $v \in R^{\binom{n}{2}}$, assume that the inequality $v.x \leqslant 0$ defines a facet of the cut cone C_n. Then the inequality $v'.x \leqslant 0$ defines a facet of the equicut polytope EP_m for all odd m, $m \geqslant 2n+1$, where $v'=(v,0,\ldots,0) \in R^{\binom{m}{2}}$.*

Proof. Set $l=m-n=n+1+2p$ with $p \geqslant 0$, $N=[1,n]$, $L=[1',l']$ and $M=N \cup L$; so C_n is the cut cone defined on the n nodes of N, EP_m is the equicut polytope defined on the m nodes of M and an equicut of EP_m is a cut $\delta(S)$ with $S \subseteq M$ and $|S|=n+p$ or $n+p+1$. Let $b.x \leqslant \beta$ be a valid inequality for EP_m such that the set $EP_m \cap \{x\colon v'.x=0\}$ is contained in the set $EP_m \cap \{x\colon b.x=\beta\}$. In order to prove that inequality $v'.x \leqslant 0$ is facet of EP_m, we must verify that there exist scalars $A>0$, B such that:

$$(*) \quad \begin{cases} b_{ij}=Av_{ij}+B & \text{for all } i<j \text{ in } N, \\ b_{ij}=B & \text{for all } i<j,\ i \in M, j \in L \text{ or } i,j \in L, \\ \beta=B(n+p)(n+p+1). \end{cases}$$

In fact, we can suppose that $\beta=0$. We prove in the next Claims 1.3–1.6 that $(*)$ holds for the inequality $b.x \leqslant 0$.

Claim 1.3. *There exists a scalar B such that $b_{ij}=B$ for all $i<j$, i,j in L.*

Proof. Take distinct points i,j,k in L and a nonzero root $\delta(S)$ of the inequality $v.x \leqslant 0$ in C_n. Set $s=|S|$, so $1 \leqslant s \leqslant n-1$, and take a subset T of $L-\{i,j,k\}$ of size $n+p-s-1$. The cuts $\delta(S \cup T \cup \{j\})$, $\delta(S \cup T \cup \{k\})$, $\delta(S \cup T \cup \{i,j\})$, $\delta(S \cup T \cup \{i,k\})$ are all equicut roots of inequality $v'.x \leqslant 0$ and thus of inequality $b.x \leqslant 0$. Using Lemma 4.2 (see Section 4), we deduce that $b_{ij}=b_{ik}$. The claim follows by symmetry. □

Claim 1.4. $B=0$.

Proof. Take a subset T of L of size $n+p+1$. The cuts $\delta(T)$ and $\delta(T-i)$ for $i \in T$ are obviously equicut roots of $v'.x \leqslant 0$ and thus of $b.x \leqslant 0$, implying

$$0=b(\delta(T))-b(\delta(T-\ i))=-b(\delta(i,T-i))+b(\delta(i,L-T))+b(\delta(i,N)).$$

Using Claim 1.3, we obtain the following relation:

$$b(\delta(i, N)) = nB. \tag{1}$$

By summing up relations (1) over $i \in T$, we obtain:

$$b(\delta(T, N)) = n(n+p+1)B. \tag{2}$$

Since $\delta(T)$ is a root, we have: $0 = b(\delta(T, M-T)) = b(\delta(T, N)) + b(\delta(T, L-T))$, which, using again Claim 1.3, implies:

$$b(\delta(T, N)) = -p(n+p+1)B. \tag{3}$$

Now, relations (2), (3) imply that $B=0$. □

Claim 1.5. *$b_{ij}=0$ for all $i \in N, j \in L$.*

Proof. Take a nonzero root $\delta(S)$ of inequality $v.x \leqslant 0$ in C_n; set $s=|S|$, so $1 \leqslant s \leqslant n-1$. Take an element $i \in L$ and subset T of $L-\{i\}$ of size $n+p-s$. Since $\delta(S \cup T)$, $\delta(S \cup T \cup \{i\})$ are both equicut roots, we obtain $0 = b(\delta(S \cup T \cup \{i\})) - b(\delta(S \cup T))$, implying that $0 = b(\delta(i, N-S)) - b(\delta(i, S)) + b(\delta(i, L-T \cup \{i\})) - b(\delta(i, T))$ and, thus, from Claims 1.3 and 1.4, $0 = b(\delta(i, N-S)) - b(\delta(i, S))$. From relation (1) and Claim 1.4, $b(\delta(i, N)) = 0$ and, therefore, $b(\delta(i, N-S)) = -b(\delta(i, S))$, implying:

$$b(\delta(S)) = 0 \tag{4}$$

Since the inequality $v.x \leqslant 0$ is facet inducing over C_n, we can find $n(n-1)/2-1$ linearly independent roots $\delta(S)$ (with $1 \notin S$) and thus (as in the proof of Theorem 1.1) we can find $n-1$ roots $\delta(S_a)$, $a=1,2,\ldots,n-1$, whose projections on the index set $\{(1,2),(1,3),\ldots,(1,n)\}$ are linearly independent. Using relation (4) applied to these $n-1$ roots $\delta(S_\alpha)$, we deduce that $b_{i2} = b_{i3} = \cdots = b_{in} = 0$ and, since $b(\delta(i, N)) = 0$, we also have $b_{i1} = 0$. □

Claim 1.6. *There exists a postive scalar A such that $b_{ij} = Av_{ij}$ for all $i<j$ in N.*

Proof. Let b_0 denote the projection of b over the index set $\{(i,j): 1 \leqslant i < j \leqslant n\}$; so, $b = (b_0, 0, \ldots, 0)$. The inequality $b_0.x \leqslant 0$ is, in fact, valid over C_n; indeed, for any cut $\delta(S)$ of C_n, take a subset T of L such that $|S \cup T| = n+p$, then $b_0(\delta(S)) = b(\delta(S \cup T)) \leqslant 0$. Also, the set $\{x \in C_n: v.x = 0\}$ is contained in the set $\{x \in C_n: b_0.x = 0\}$; from the assumption that the inequality $v.x \leqslant 0$ is facet inducing over C_n and the fact that C_n is full dimensional, we deduce that $b_0 = Av$ for some positive scalar A. □

2. Complete description of some restricted cones and polytopes

In this section, we give several results permitting, in particular, to obtain the complete description of the inequicut and equicut cones and polytopes for $n \leqslant 5$. We first state a lemma about linear combinations of cut vectors.

Lemma 2.1. *Given the vector* $x=\sum_{S\subseteq[2,n]}\lambda_S\chi^{\delta(S)}$ *with* $\lambda_S\in R$ *for all* $S\subseteq[2,n]$, *the following assertions hold*:

(i) $\lambda_{i,j}=\frac{1}{2}(x_{1i}+x_{1j}-x_{ij})-\sum_{S\subseteq[2,n],|S|\geqslant 3}\lambda_S$ for $2\leqslant i<j\leqslant n$.

(ii) $\lambda_i=-x_{1i}+\sum_{2\leqslant j\leqslant n, j\neq i}\frac{1}{2}(x_{1i}+x_{1j}-x_{ij})+\sum_{\{i\}\subseteq S,|S|\geqslant 3}\lambda_S(|S|-2)$ for $2\leqslant i\leqslant n$.

(iii) $\displaystyle\sum_{S\subseteq[2,n]}\lambda_S=\frac{1}{2}\left(\sum_{1\leqslant i<j\leqslant n}x_{ij}-(n-3)\sum_{2\leqslant i\leqslant n}x_{1i}\right)+\sum_{S\subseteq[2,n],|S|\geqslant 3}\binom{|S|-1}{2}\lambda_S.$

The proof is an easy verification.

As an application, we obtain the complete description of the equicut cone EC_4 and of the equicut polytope EP_4. Given a vector $x\in R^6$, x belongs to EC_4 if and only if

$$x=\sum_{S\subseteq[2,4]}\lambda_S\chi^{\delta(S)}$$

with $\lambda_S\geqslant 0$ for $|S|=2$ and $\lambda_S=0$ for $|S|\neq 2$; in this case, the above relations (i) and (ii) become

$$\lambda_{23}=\tfrac{1}{2}(x_{12}+x_{13}-x_{23})\geqslant 0,$$

$$\lambda_{24}=\tfrac{1}{2}(x_{12}+x_{14}-x_{24})\geqslant 0,$$

$$\lambda_{34}=\tfrac{1}{2}(x_{13}+x_{14}-x_{34})\geqslant 0,$$

$$\lambda_2=-x_{12}+\sum_{j=3,4}\frac{x_{12}+x_{1j}-x_{2j}}{2}=0,$$

$$\lambda_3=-x_{13}+\sum_{j=2,4}\frac{x_{13}+x_{1j}-x_{3j}}{2}=0,$$

$$\lambda_4=-x_{14}+\sum_{j=2,3}\frac{x_{14}+x_{1j}-x_{4j}}{2}=0.$$

Therefore,

$$EC_4=\{x\in R^6\colon\ x_{12}+x_{13}-x_{23}\geqslant 0,\ x_{12}+x_{14}-x_{24}\geqslant 0,\ x_{13}+x_{14}-x_{34}\geqslant 0 \text{ and }$$
$$x_{13}+x_{14}+x_{34}=x_{23}+x_{24}+x_{34}=x_{12}+x_{14}+x_{24}=x_{23}+x_{24}+x_{34}\}.$$

Using the relation (iii) $\lambda_{23}+\lambda_{24}+\lambda_{34}=\frac{1}{2}(x_{23}+x_{24}+x_{34})$ and the additional condition $\sum\lambda_S=1$ for vectors $x=\sum\lambda_S\lambda^{\delta(S)}$ of the equicut polytope EP_4, we deduce that:

$$EP_4=EC_4\cap\{x\in R^6\colon\ x_{23}+x_{24}+x_{34}=2\}.$$

One could deduce similarly the description of the inequicut cone and polytope IC_n, IP_n for $n=4,5$; such descriptions will be, in fact, a byproduct of Lemmas 2.2 and 2.3.

We now consider the k-uniform cut cone C_n^k for $k=1, 2$; C_n^1 is generated by the cut vectors $\chi^{\delta(\{i\})}$ for $1 \leqslant i \leqslant n$ and C_n^2 is generated by the cut vectors $\chi^{\delta(\{i,j\})}$ for $1 \leqslant i < j \leqslant n$. Similarly, the polytopes P_n^1, P_n^2 are defined as the convex hulls of the cut vectors $\chi^{\delta(\{i\})}$ for $1 \leqslant i \leqslant n$, $\chi^{\delta(\{i,j\})}$ for $1 \leqslant i < j \leqslant n$, respectively. The family $\{\chi^{\delta(\{i,j\})}: 1 \leqslant i < j \leqslant n\}$ is linearly independent (see (6) below); hence both cones C_n^1, C_n^2 are simplicial of dimension n, $\binom{n}{2}$, respectively, and the polytopes P_n^1, P_n^2 have dimension $n-1$, $\binom{n}{2}-1$, respectively.

Lemma 2.2. (i) $C_n^1 = \{x \in R^{\binom{n}{2}}: x_{12}+x_{13}-x_{23} \geqslant 0$ *and* $x_{1i}+x_{1j}-x_{ij} = x_{12}+x_{13}-x_{23}$ *for all* $2 \leqslant i < j \leqslant n\}$,

(ii) $P_n^1 = C_n^1 \cap \{x \in R^{\binom{n}{2}}: \sum_{1 \leqslant i < j \leqslant n} x_{ij} = n-1\}$.

Proof. Take $x \in C_n^1$, then $x = \sum_{i=1}^n \lambda_i \chi^{\delta(i)}$ with $\lambda_i \geqslant 0$ and, hence, $x_{ij} = \lambda_i + \lambda_j$ for all $1 \leqslant i < j \leqslant n$; this implies the relations:

(a) $\lambda_1 = \frac{1}{2}(x_{1i}+x_{1j}-x_{ij})$ for all $2 \leqslant i < j \leqslant n$,

(b) $\lambda_i = \frac{1}{2}(x_{1i}+x_{ij}-x_{1j})$ for all $j \neq 1, i$,

henceforth, yielding: $x_{1i}+x_{1j}-x_{ij} = x_{12}+x_{13}-x_{23} \geqslant 0$ for all $2 \leqslant i < j \leqslant n$. Conversely, take x such that $x_{1i}+x_{1j}-x_{ij} = x_{12}+x_{13}-x_{23} \geqslant 0$ for all $2 \leqslant i < j \leqslant n$. Then $x_{1i}+x_{1j}-x_{ij}$ does not depend on the choice of j, $j \neq 1, i$. We verify that $x = \sum_{i=1}^n \lambda_i \chi^{\delta(\{i\})}$ with $\lambda_1, \ldots, \lambda_n$ defined by above relations (a) and (b); indeed $x_{ij} = \lambda_i + \lambda_j = \frac{1}{2}(x_{1i}+x_{ij}-x_{1j}) + \frac{1}{2}(x_{1j}+x_{ij}-x_{1i})$. This states the assertion (i). In order to prove (ii), one needs only observe that the only cut vectors satisfying $\sum x_{ij} = n-1$ are the cut vectors $\chi^{\delta(\{i\})}$, $1 \leqslant i \leqslant n$. □

We assume that $n \geqslant 5$.

For $1 \leqslant i < j \leqslant n$, we define the linear form:

$$L_{ij}(x) = 2 \sum_{1 \leqslant h < k \leqslant n} x_{hk} + (n-2) \sum_{\substack{1 \leqslant k \leqslant n \\ k \neq i,j}} (x_{ij} - x_{ik} - x_{jk}) - 4(n-2)x_{ij}.$$

Lemma 2.3. *For* $n \geqslant 5$,

(i) $C_n^2 = \{x \in R^{\binom{n}{2}}: L_{ij}(x) \leqslant 0$ for $1 \leqslant i < j \leqslant n\}$,

(ii) $P_n^2 = C_n^2 \cap \{x \in R^{\binom{n}{2}}: \quad L(x) = -2(n-2)(n-4)\}$,

where $L(x) = \sum_{1 \leqslant i < j \leqslant n} L_{ij}(x) = -(n-4) \sum_{1 \leqslant i < j \leqslant n} x_{ij}$.

Proof. It can easily be verified that:

$$L_{ij}(\chi^{\delta(h,k\})}) = \begin{cases} -2(n-2)(n-4) & \text{if } (h,k) = (i,j), \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

An easy consequence of this fact is that

$$\text{the family } \{\chi^{\delta(\{i,j\})}: 1 \leqslant i < j \leqslant n\} \text{ is linearly independent.} \tag{6}$$

Therefore, any vector $x\in R^{\binom{n}{2}}$ admits a decomposition: $x=\sum_{1\leqslant i<j\leqslant n}\lambda_{ij}\chi^{\delta(\{i,j\})}$ for $\lambda_{ij}\in R$ and, from relation (5), $\lambda_{ij}=-L_{ij}(x)/2(n-2)(n-4)$ for all $1\leqslant i<j\leqslant n$. Consequently, $x\in C_n^2$ if and only if $L_{ij}(x)\leqslant 0$ for all $1\leqslant i<j\leqslant n$. Hence, the family of inequalities $L_{ij}(x)\leqslant 0$ for $1\leqslant i<j\leqslant n$ includes the family of the facets of C_n^2 and, in fact, coincides with it since C_n^2 is a simplicial cone. Furthermore, $x\in P_n^2$ if and only if $x=\sum_{1\leqslant i<j\leqslant n}\lambda_{ij}\chi^{\delta(\{i,j\})}$ with $\lambda_{ij}\geqslant 0$ and $\sum\lambda_{ij}=1$; therefore, $x\in P_n^2$ if and only if $x\in C_n^2$ and $L(x)=\sum_{1\leqslant i<j\leqslant n}L_{ij}(x)=-2(n-2)(n-4)$. □

It is also easy to characterize the points $x\in Z^{\binom{n}{2}}$ of L_n^2, the lattice generated by the cut vectors $\chi^{\delta(\{i,j\})}$, i.e., the points which are linear combinations with integer coefficients of the cut vectors $\chi^{\delta(\{i,j\})}$. Indeed, from (5), $x\in L_n^2$ if and only if $2(n-2)(n-4)$ divides $L_{ij}(x)$ for all $1\leqslant i<j\leqslant n$ or, equivalently, if and only if $2(n-2)(n-4)$ divides

$$L_{ij}(x)-2(n-2)(n-4)x_{ij}=2\sum_{1\leqslant h<k\leqslant n}x_{hk}-(n-2)\sum_{\substack{1\leqslant k\leqslant n\\ k\neq i,j}}(x_{ij}+x_{ik}+x_{jk})$$

for all $1\leqslant i<j\leqslant n$. For instance, one obtains that, for $n=5, x\in L_5^2$ if and only if 3 divides $\sum_{1\leqslant h<k\leqslant 5}x_{hk}$ and 2 divides $x_{ij}+x_{ik}+x_{jk}$ for all $1\leqslant i<j\leqslant 5$.

Remark. Consider now the polytopes $\bar{P}_n^1$, $\bar{P}_n^2$ defined as the convex hulls of the cut vectors $\chi^{\delta(\{i\})}$ together with the zero vector, the cut vectors $\chi^{\delta(\{i,j\})}$ together with the zero vector, respectively. One obtains that:

$$\bar{P}_n^1=C_n^1\cap\left\{x\in R^{\binom{n}{2}}:\ \sum_{1\leqslant i<j\leqslant n}x_{ij}\leqslant n-1\right\},$$

$$\bar{P}_n^2=C_n^2\cap\{x\in R^{\binom{n}{2}}:\ L(x)\leqslant -2(n-2)(n-4)\}.$$

As a corollary of the above Lemmas 2.2 and 2.3, we deduce the description of IC_n, EP_n for $n=4, 5$. Indeed, we have $IC_4=C_4^1$, $IC_5=C_5^1$, $EC_5=C_5^2$ and $EP_5=P_5^2$. More precisely, we obtain that:

$$IC_4=\{x\in R^6:\ x_{12}+x_{13}-x_{23}=x_{12}+x_{14}-x_{24}=x_{13}+x_{14}-x_{34}\geqslant 0\},$$

$$IC_5=\{x\in R^{10}:\ x_{12}+x_{13}-x_{23}=x_{1i}+x_{1j}-x_{ij}\geqslant 0 \text{ for } 1\leqslant i<j\leqslant 5\},$$

$$EP_5=\{x\in R^{10}: x_{ij}+x_{ik}+x_{jk}\leqslant 2 \text{ for } 1\leqslant i<j<k\leqslant 5 \text{ and } \textstyle\sum_{1\leqslant i<j\leqslant 5}x_{ij}=6\}.$$

We conclude this section with some observations on the intersection $IC_n\cap EP_n$. For $n=3$, $IC_3=\emptyset$ and $IC_n\cap EP_n$ is nonempty for $n\geqslant 4$. As an example, we describe this intersection for the simplest case: $n=4$. If $x\in IC_4\cap EP_4$, then $x=\lambda_1\chi^{\delta(\{1\})}+\lambda_2\chi^{\delta(\{2\})}+\lambda_3\chi^{\delta(\{3\})}+\lambda_4\chi^{\delta(\{4\})}=\mu_{23}\chi^{\delta(\{2,3\})}+\mu_{24}\chi^{\delta(\{2,4\})}+\mu_{34}\chi^{\delta(\{3,4\})}$ with $\lambda_i\geqslant 0$, $\mu_{ij}\geqslant 0$, $\mu_{23}+\mu_{24}+\mu_{34}=1$ and, for instance, $\lambda_1\neq 0$. Using relation: $\chi^{\delta(\{1\})}+\chi^{\delta(\{2\})}+\chi^{\delta(\{3\})}+\chi^{\delta(\{4\})}=\chi^{\delta(\{2,3\})}+\chi^{\delta(\{2,4\})}+\chi^{\delta(\{3,4\})}$ and the fact that the family $\{\chi^{\delta(\{2\})},\chi^{\delta(\{3\})},\chi^{\delta(\{4\})},\chi^{\delta(\{2,3\})},\chi^{\delta(\{2,4\})},\chi^{\delta(\{3,4\})}\}$ is linearly independent, we deduce that: $\lambda_1=\lambda_2=\lambda_3=\lambda_4=\mu_{23}=\mu_{24}=\mu_{34}=\frac{1}{3}$, i.e. $x=\frac{2}{3}(\chi^{\delta(\{2,3\})}+\chi^{\delta(\{2,4\})}+\chi^{\delta(\{3,4\})})=\frac{2}{3}\mathbf{1}$, where $\mathbf{1}=(1,1,1,1,1,1)$. Therefore, we have that: $IC_4\cap EP_4=\{\frac{2}{3}\mathbf{1}\}$ and, furthermore, $IC_4\cap EC_4=R_+1$.

3 Some classes of facets for the inequicut cone IC_n

In this section, we describe several classes of facets for the inequicut cone IC_n. Actually, our starting point was the list of facets of IC_6 and IC_7 for which we tried to find an extension of IC_n for any n.

We obtained the description of the facets of IC_n for $n=6, 7$ by computer. Since the computation was done approximately with (finite digit) floating point arithmetic, there is no guarantee that the results are correct or complete. However, we are convinced that the list is complete for $n=6, 7$; also, an important fact supporting this conclusion is that, in the computer output, each of the described facets was listed together with all its permutation equivalent. Later, V. Grishukhin (personal communication) confirmed our results for $n=6, 7$ by a similar computation. From the point of view of optimization, the study of facets for small n may not look very interesting, but it is actually the inspection of the small values which enabled us to obtain several classes of facets for general n. Examples of other works dealing with small values are [1] (for $n=6, 7$), [6] (for $n=8$) on small traveling salesman polytopes, [10] on small multicut polytopes (for $n=4, 5$).

We present in Sections 3.1 and 3.2 the facets of IC_6 and IC_7, respectively; IC_6 has, up to permutation, five types of facets, denoted as (3.1a)–(3.5a) and IC_7 has 10 types of facets denoted as (3.1b)–(3.10b).

We succeeded to find a generalization, for arbitrary even n, of all the facets of IC_6 except for the facet (3.5a). The facets (3.2a), (3.3a) and (3.4a) belong, respectively, to the general classes of gyroscope inequalities Gyr_{2p}, domino inequalities Dom_{2p} and butterfly inequalities But_{2p}, that we describe below. The inequality (3.1a) or (3.1b) is simply the known triangle inequality for the cut polytope and, therefore, from Theorem 1.1, it is facet inducing for IC_n for any $n>7$. By confronting the known facets of IC_6, IC_7, we noticed that each facet of IC_6 has its 'odd' analogue for IC_7; namely, the analogue of facet (3.ia) is facet (3.ib) for $i=1,2,3,4,5$. We could find an analogue to the odd case for the classes of gyroscope and domino inequalities; so, the facets (3.2b) and (3.3b) belong, respectively, to the classes of gyroscope inequalities Gyr_{2p+1} and domino inequalities Dom_{2p+1}. The analogue of facet (3.4a) is facet (3.4b) for $n=7$. However, we do not have a satisfying generalization of (3.4b) for any odd $n\geqslant 9$.

We present below the new classes of facets for IC_n, namely, the class of gyroscope inequalities in Section 3.3, the class of domino inequalities in Section 3.4 and the class of butterfly inequalities in Section 3.5. Finally, in Section 3.6, we consider the equality case; namely, for each class of facets, we give the explicit description of its roots. We also remark that each class of facets can be triangulated, i.e. decomposed as a linear combination of triangles.

3.1. Description of IC_6

We give the complete description (up to permutation) of the facets of IC_6, namely, they are induced by the inequalities (3.1a)–(3.5a) below:

(3.1a) *Triangle facet*

$$x_{ij}-x_{ik}-x_{jk}\leqslant 0 \quad \text{for distinct } i,j,k\in\{1,\ldots,6\}.$$

There are $\binom{6}{3}\binom{3}{1}=60$ permutation equivalent such facets.

(3.2a) *Gyroscope facet*: Gyr_6

$$x_{14}+x_{23}-x_{13}-x_{24}-\sum_{i=5,6}(x_{1i}+x_{2i})+2x_{56}\leqslant 0.$$

There are $\binom{6}{2}\binom{4}{2}\binom{2}{1}=180$ permutation equivalent such facets.

(3.3a) *Domino facet*: Dom_6

$$2(x_{12}+x_{34}+x_{56})-\sum_{i=1,2,5,6}(x_{3i}+x_{4i})\leqslant 0.$$

There are $\frac{1}{2}\binom{6}{2}\binom{4}{2}=45$ permutation equivalent such facets.

(3.4a) *Butterfly facet*: But_6

$$\sum_{i=1,2}(x_{4i}-x_{3i})+\sum_{i=5,6}(x_{3i}-x_{4i})-2x_{34}\leqslant 0.$$

There are $\binom{6}{2}\binom{4}{2}=90$ permutation equivalent such facets.

(3.5a)

$$2(x_{12}+x_{23}+x_{13}+x_{45})+x_{56}-\sum_{i=1,2,3,6}x_{4i}-3\sum_{i=1,2,3}x_{5i}\leqslant 0.$$

There are $\binom{6}{3}\binom{3}{1}\binom{2}{1}=120$ permutation equivalent such facets.

All these facets except Gyr_6 were first found by Avis (personal communication) by computer. In Sections 3.3–3.5, we describe a generalization of the facets (3.2a)–(3.4a) to the case of any even $n\geqslant 6$. The only simplicial facet is facet (3.5a). So, altogether, the inequicut cone IC_6 has 495 facets.

We show in Figs. 3–5 the supporting graph for the facets (3.2a) Gyr_6, (3.3a) Dom_6, (3.4a) But_6, respectively. In all figures, the edges associated with the coefficient $+1$ (or -1) are indicated by single solid (or dotted) lines, and the edges with the coefficient $+2$ (or -2) are indicated by double solid (or dotted) lines, etc.

3.2. Description of IC_7

Up to permutation, all the facets of the inequicut cone IC_7 are induced by the inequalities (3.1b)–(3.10b) below.

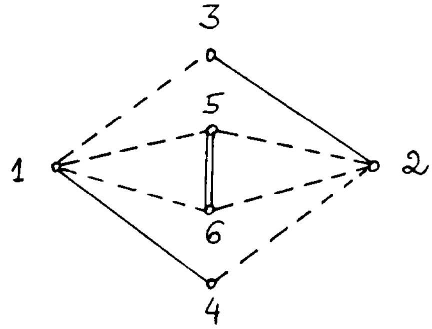

Fig. 3. Gyr_6

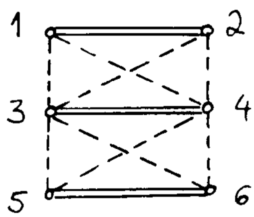

Fig. 4. Dom_6

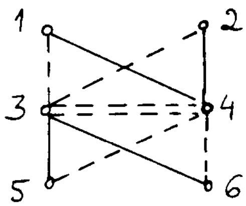

Fig. 5. But_6

(3.1b) *Triangle facet*:

$$x_{ij}-x_{ik}-x_{jk}\leqslant 0 \quad \text{for all distinct } i,j,k \text{ in } \{1,\ldots,7\}.$$

There are $\binom{7}{3}\binom{3}{1}=105$ permutation equivalent such facets.

(3.2b) *Gyroscope facet*: Gyr_7

$$x_{12}-x_{13}+x_{14}+x_{23}-x_{24}+x_{56}+x_{57}+x_{67}-\sum_{i=5,6,7}(x_{1i}+x_{2i})\leqslant 0.$$

There are $\binom{7}{3}\binom{4}{2}\binom{2}{1}=420$ permutation equivalent such facets.

(3.3b) *Domino facet*: Dom_7

$$2x_{12}+3x_{34}+x_{56}+x_{57}+x_{67}-\sum_{i=3,4}(x_{1i}+x_{2i}+x_{5i}+x_{6i}+x_{7i})\leqslant 0.$$

There are $\binom{7}{3}\binom{4}{2}=210$ permutation equivalent such facets.

(3.4b) *Butterfly facet*: But_7

$$\sum_{i=1,2,3}(x_{4i}-x_{5i})-3x_{45}-2\sum_{i=6,7}(x_{4i}-x_{5i})\leqslant 0.$$

There are $\binom{7}{1}\binom{6}{1}\binom{5}{2}=420$ permutation equivalent such facets.

(3.5b)

$$3(x_{12}+x_{13}+x_{23})-\sum_{i=1,2,3}(4x_{4i}+x_{5i})+3x_{45}+\sum_{i=6,7}(x_{4i}-x_{5i})\leqslant 0.$$

There are $\binom{7}{3}\binom{4}{2}\,2=420$ permutation equivalent such facets.

(3.6b)

$$x_{12}+x_{23}+x_{34}+x_{45}+x_{15}-\sum_{1\leqslant i\leqslant 5}(x_{6i}+x_{7i})+3x_{67}\leqslant 0.$$

There are $\frac{1}{2}\binom{7}{2}\,4!=252$ permutation equivalent such facets.

(3.7b)

$$2(x_{15}+x_{24})-(x_{13}+x_{14}+x_{23}+x_{25})-2(x_{34}+x_{35})$$
$$+\sum_{i=6,7}(2x_{3i}-x_{4i}-x_{5i})\leq 0.$$

There are $\binom{7}{2}\binom{5}{2}6=1260$ permutation equivalent such facets.

(3.8b)

$$\sum_{i=1,2}(x_{4i}-x_{5i})-2(x_{34}-x_{35})-\sum_{i=6,7}(2x_{4i}+x_{5i})+3x_{67}\leq 0.$$

There are $\binom{7}{2}\binom{5}{2}6=1260$ permutation equivalent such facets.

(3.9b)

$$4x_{12}-\sum_{i=1,2}(2x_{3i}+x_{4i}+x_{5i}+x_{6i}+x_{7i})+2x_{34}+2x_{35}-x_{36}-x_{37}+3x_{67}\leq 0.$$

There are $\binom{7}{1}\binom{6}{2}\binom{4}{2}=630$ permutation equivalent such facets.

(3.10b)

$$3x_{12}+3x_{34}-\sum_{i=1,2,3,4}(2x_{5i}+x_{6i})-3x_{56}+x_{57}-x_{67}\leq 0.$$

There are $\frac{1}{2}\binom{7}{4}\binom{4}{2}3\cdot 2=630$ permutation equivalent such facets.

In the above list, the simplicial facets are (3.2b), (3.5b), (3.6b), and (3.7b). So, altogether, IC_7 has 5607 facets and 2352 of them are simplicial. In Sections 3.3 and 3.4, we give a generalization of the facets (3.2b) Gyr_7 and (3.3b) Dom_7 for the case of any odd $n\geq 7$.

We conclude with the description of the inequicut polytope IP_n for $n=6,7$. Up to permutation, the full description of IP_6 (or IP_7) is obtained by taking all the five facets of IC_6 (or all the 10 facets of IC_7) together with the following three facets (the first two of them are simplicial).

For $n=6$, $Q(1,1,1,1,1,1).x\leq 8$, $\quad Q(-2,1,1,1,1,1).x\leq 2$,
$Q(1,1,0,0,0,0).x+Q(0,0,1,1,1,1).x+Q(-1,-1,1,1,1,1).x\leq 4$.

For $n=7$, $Q(1,1,1,1,1,1,1).x\leq 10$, $\quad Q(-3,1,1,1,1,1,1).x\leq 2$,
$2Q(1,1,0,0,0,0,0).x+Q(-1,-1,1,1,1,1,1).x\leq 2$.

(Recall that $Q(b_1,\ldots,b_n).x$ denotes the quantity: $\sum_{1\leq i<j\leq n}b_ib_jx_{ij}$.)

3.3. The gyroscope inequality Gyr_n for $n\geq 6$

In the even case $n=2p$, $p\geq 3$, the gyroscope inequality, denoted as: $Gyr_{2p}.x\leq 0$, is the inequality:

$$x_{14}+x_{23}-x_{13}-x_{24}-\sum_{2\leq i\leq p-1}(x_{1,2i+1}+x_{1,2i+2}+x_{2,2i+1}+x_{2,2i+2}$$
$$-2x_{2i+1,2i+2})\leq 0.$$

For instance, the number of permutation equivalent facets to Gyr_{2p} is equal to:

$$\binom{2p}{4}\binom{4}{2}\binom{2}{1}\cdot\prod_{i=2}^{p-1}\binom{2p-2i}{2}=\frac{(2p)!}{2^{p-1}}.$$

In the odd case $n=2p+1$, $p\geqslant 3$, the gyroscope inequality is the inequality $Gyr_{2p+1}.x\leqslant 0$ described by:

$$x_{12}+x_{14}+x_{23}-x_{13}-x_{24}+x_{2p-1,2p}+x_{2p,2p+1}+x_{2p-1,2p+1}$$

$$-\sum_{2\leqslant i\leqslant p-2}(x_{1,2i+1}+x_{1,2i+2}+x_{2,2i+1}+x_{2,2i+2}$$

$$-2x_{2i+1,2i+2})-\sum_{i=2p-1,2p,2p+1}(x_{1i}+x_{2i})\leqslant 0.$$

Theorem 3.1. *The gyroscope inequality $Gyr_n.x\leqslant 0$ defines a facet of the inequicut cone IC_n for all $n\geqslant 6$.*

The proof is given in Section 4.1. We show in Figs. 6 and 7 the supporting graphs of Gyr_{2p} and Gyr_{2p+1}, respectively.

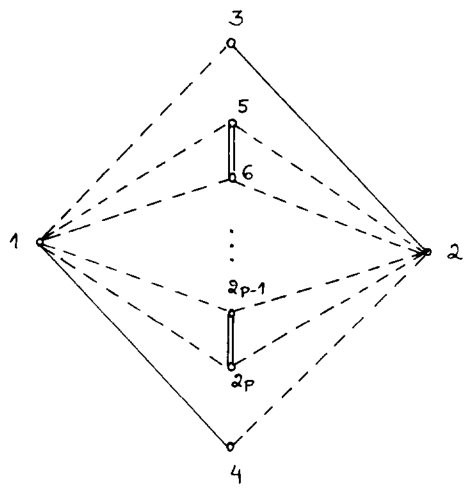

Fig. 6. Gyr_{2p}

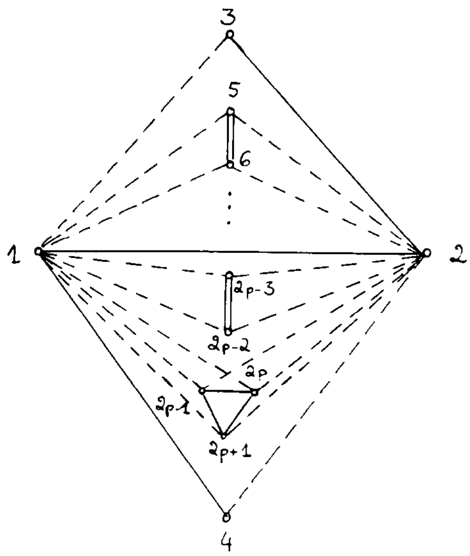

Fig. 7. Gyr_{2p+1}

3.4. The domino inequality Dom_n for $n \geqslant 6$

In the even case $n=2p$, $p \geqslant 3$, the domino inequality is the inequality $Dom_{2p}.x \leqslant 0$ defined by:

$$\sum_{0 \leqslant i \leqslant p-2} (2x_{2i+1,2i+2} - x_{2i+1,2i+3} - x_{2i+1,2i+4} - x_{2i+2,2i+3} - x_{2i+2,2i+4})$$
$$+ 2x_{2p-1,2p} \leqslant 0.$$

In the odd case, $n=2p+1$, $p \geqslant 3$, the domino inequality is the inequality $Dom_{2p+1}.x \leqslant 0$ defined by:

$$\sum_{0 \leqslant i \leqslant p-2} (2x_{2i+1,2i+2} - x_{2i+1,2i+3} - x_{2i+1,2i+4} - x_{2i+2,2i+3} - x_{2i+2,2i+4})$$
$$+ x_{2p-3,2p-2} - x_{2p-3,2p+1} - x_{2p-2,2p+1} + x_{2p-1,2p+1} + x_{2p,2p+1}$$
$$+ x_{2p-1,2p} \leqslant 0.$$

Theorem 3.2. *The domino inequality $Dom_n.x \leqslant 0$ induces a facet of the inequicut cone IC_n for all $n \geqslant 6$.*

The proof is given in Section 4.2. We show in Figs. 8 and 9 the supporting graphs of Dom_{2p} and Dom_{2p+1}, respectively.

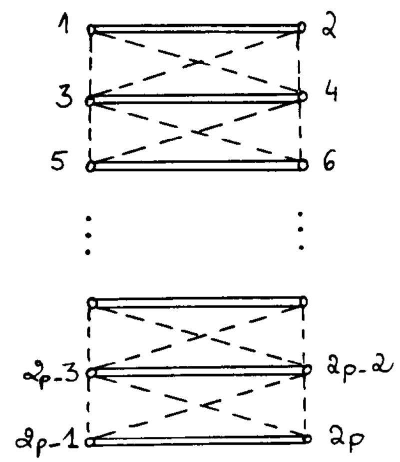

Fig. 8. Dom_{2p}

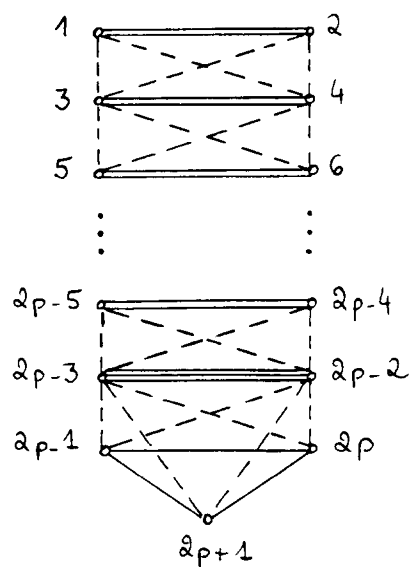

Fig. 9. Dom_{2p+1}

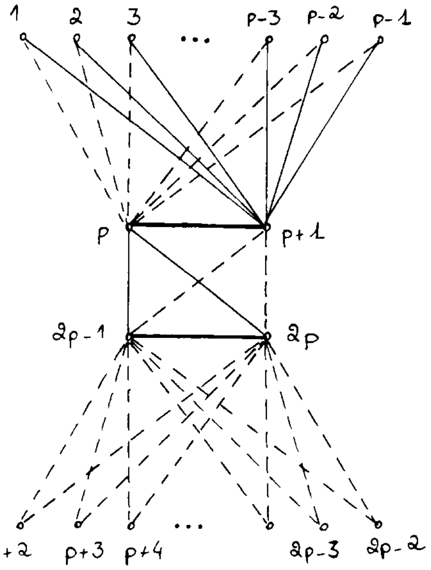

Fig. 10. But_{2p} (the edges $(p,p+1),(2p-1,2p)$ have weights $-(p-1),p-3$, respectively)

3.5. *The butterfly inequality But_{2p} for $p\geqslant 3$*

For even $n=2p$, $p\geqslant 3$, the butterfly inequality is the inequality: $But_{2p}.x\leqslant 0$ described by:

$$\sum_{1\leqslant i\leqslant p-1}(-x_{ip}+x_{i,p+1})-\sum_{p+2\leqslant i\leqslant 2p-2}(x_{i,2p-1}+x_{i,2p})+x_{p,2p-1}+x_{p,2p}$$
$$-x_{p+1,2p-1}-x_{p+1,2p}-(p-1)x_{p,p+1}+(p-3)x_{2p-1,2p}\leqslant 0.$$

Theorem 3.3. *The butterfly inequality $But_{2p}.x\leqslant 0$ induces a facet of the inequicut cone IC_{2p} for all $p\geqslant 3$.*

The proof is given in Section 4.3. We show in Fig. 10 the supporting graph of But_{2p}. Note that the butterfly inequality is violated only by the equicut $\delta([1,p])$.

3.6. *The equality case for Gyr_n, Dom_n, But_{2p}*

In this section, we collect some additional information on the new classes of facets; in particular, we consider the equality case, i.e. we describe their roots (the inequicuts satisfying the given inequality at equality). A good understanding of the roots is indeed useful for proving that the inequality at hand is facet inducing.

A nice feature common to the gyroscope, domino and butterfly inequalities is that they can be triangulated, i.e. they can be expressed as linear combinations of triangle inequalities (including degenerate ones). More precisely, denoting by

$T.x := x_{ij} - x_{ik} - x_{jk} \leqslant 0$ a triangle inequality, it is said to be degenerate if the nodes i, j coincide, i.e. if it is of the form $-2x_{ik} \leqslant 0$; then, the inequality $v.x \leqslant 0$ can be triangulated if $v.x$ can be decomposed as $v.x. = \sum_a \lambda_a T_a.x$, where $\lambda_a \in R$, T_a are some triangles. Explicit triangulations of the gyroscope, domino and butterly inequalities are given in Section 4, namely, in relations (7), (13), (18), (23) and (27). An interesting consequence of this property is that $Gyr_n.\chi^{\delta(S)}$, $Dom_n.\chi^{\delta(S)}$, $But_{2p}.\chi^{\delta(S)}$ are even integers for any cut $\delta(S)$. Actually, as can be seen through the proofs in Section 4, the explicit decomposition of Gyr_n, Dom_n, But_{2p} as sum of triangles is a crucial tool for checking their validity for the inequicut cone and, then, for identifying their roots.

We now give the description of the roots for each of the inequalities Gyr_{2p}, Gyr_{2p+1}, Dom_{2p}, Dom_{2p+1}, But_{2p}.

For this, we introduce some terminology which should help describing the roots in a more pictorial way. Given i odd in $[1, 2p]$, a subset S of $[1, 2p]$ is called *mixed at level i* if $|S \cap \{i, i+1\}| = 1$ and *sparse at level i* if $|S \cap \{i, i+1\}| \leqslant 1$.

The inequicut roots of the gyroscope inequality $Gyr_{2p}.x \leqslant 0$ are the cuts $\delta(S)$ for which $S \subseteq [1, 2p]$ is of type (A1), (A2) or (A3). (Of course, the complement of S defines the same root as S.)

(A1) $S \cap \{1,2,3,4\} = \{1\}$ *or* $\{2\}$ and S is mixed at level i for all odd i with $5 \leqslant i \leqslant 2p-1$.

(A2) $S \cap \{1,2,3,4\} = \emptyset$, $\{3\}$, $\{4\}$ *or* $\{3,4\}$ and S is sparse at level i for all odd i with $5 \leqslant i \leqslant 2p-1$.

(A3) $S \cap \{1,2,3,4\} = \{1,3\}$ *or* $\{2,4\}$ and S is mixed at level i for all but one (say odd j) odd i with $5 \leqslant i \leqslant 2p-1$ and $S \cap \{j, j+1\} = \emptyset$.

We now describe the inequicut roots of $Gyr_{2p+1}.x \leqslant 0$ using, in particular, the above description of the classes (A1)–(A3) of roots of $Gyr_{2p}.x \leqslant 0$. The roots of $Gyr_{2p+1}.x \leqslant 0$ are the cuts $\delta(S)$ for which $S \subseteq [1, 2p+1]$ is of type (A1), (A3) or (A'1), (A'2), (A'3), (A'4).

(A'1) S is of type (A2), except $|S| = p$ (i.e. $S \cap \{1,2,3,4\} = \{3,4\}$ and S is mixed at level i for all odd i, $5 \leqslant i \leqslant 2p-1$).

(A'2) $S = S_1 \cup \{2p+1\}$, where S_1 is of type (A2) and $S_1 \cap \{2p-1, 2p\} = \emptyset$, except $|S_1| = p-1$ (i.e. $S_1 \cap \{1,2,3,4\} = \{3,4\}$ and S is mixed at all odd levels $i, i \geqslant 5$).

(A'3) $S \cap \{1,2,3,4\} = \{1,3\}$ *or* $\{2,4\}$ and S is mixed at level i for all odd i with $5 \leqslant i \leqslant 2p-3$ and $i \neq j$, where j is some fixed odd integer between 5 and $2p-3$, and $j, j+1, 2p-1, 2p$ belong to S.

(A'4) $S \cap \{1,2,3,4\} = \{1,3,4\}$ *or* $\{2,3,4\}$, S is mixed at level i for all odd i with $5 \leqslant i \leqslant 2p-3$ and $2p-1, 2p$ belong to S.

The inequicut roots of the domino inequality $Dom_{2p}.x \leqslant 0$ are the cuts $\delta(S)$, where $S \subseteq [1, 2p]$ is of the type (B).

(B) S is mixed at level i for all odd i with $i \in [1, i_1] \cup [i_2, 2p-1]$, $S \cap \{i, i+1\} = 0$ for $i \in [i_1+2, i_2-2]$, where i_1, i_2 are odd integers satisfying $1 \leqslant i_1$, $i_2 \leqslant 2p-1$ and $i_2 \geqslant i_1 + 4$.

The inequicut roots of $Dom_{2p+1}.x \leqslant 0$ are the cuts $\delta(S)$ for which $S \subseteq [1, 2p+1]$ is of the type (B), (B′1), (B′2).

(B′1) $S = S_1 \cup \{2p+1\}$, where S_1 is of type (B) and $|S \cap \{2p-3, 2p-2\}| = |S \cap \{2p-1, 2p\}|$.

(B′2) S is mixed at level i for all odd i such that $i_1 \leqslant i \leqslant 2p-3$, $|S \cap \{2p-1, 2p\}| = 2$, $|S \cap \{i, i+1\}| = e$ for all odd i with $1 \leqslant i \leqslant i_1 - 2$, where i_1 is an odd integer with $3 \leqslant i_1 \leqslant 2p-3$ and $e = 0$ or 2.

The inequicut roots of the butterfly inequality $But_{2p}.x \leqslant 0$ are the cuts $\delta(S)$ for which $S \subseteq [1, 2p]$ is of one of the following types (C1)–(C6).

(C1) S is a subset of $[1, p-1]$.

(C2) $S = S_1 \cup \{p+1, 2p-1\}$ or $S_1 \cup \{p+1, 2p\}$, where S_1 is a subset of $[p+2, 2p-2]$.

(C3) $S = S_1 \cup S_2 \cup \{2p-1\}$ or $S_1 \cup S_2 \cup \{2p\}$, where S_1 is a subset of $[1, p-1]$, S_2 is a subset of $[p+2, 2p-2]$ with $|S_1 \cup S_2| \neq p-1$.

(C4) $S = S_1 \cup \{p, p+1\}$, where S_1 is a subset of $[1, p-1]$ with $|S_1| \neq p-2$.

(C5) $S = [1, p] \cup \{i\}$ with $p+2 \leqslant i \leqslant 2p-2$.

(C6) $S = [1, p] - \{i\}$ with $1 \leqslant i \leqslant p-1$.

4. Proofs

In this section, we give the proofs of Theorems 3.1–3.3 stating that the gyroscope, domino and butterfly inequalities are facet inducing. The proofs are based on the polyhedral method; namely, in order to show that a given valid inequality $v.x \leqslant 0$ defines a facet of IC_n, we take a valid inequality $b.x \leqslant 0$ for IC_n such that $\{x \in IC_n : v.x = 0\} \subseteq \{x \in IC_n : b.x = 0\}$ and we prove that $b = \alpha v$ for some positive scalar α. We first state two lemmas that will be thoroughly used in the proofs; they follow, respectively, from [5, 4, Lemma 2.5].

Lemma 4.1. Let b a vector in $R^{\binom{n}{2}}$. Let i, j be distinct elements of $[1, n]$ and S be a subset (possibly empty) of $[1, n] - \{i, j\}$ such that the cut vectors $\chi^{\delta(S)}$, $\chi^{\delta(S \cup \{i\})}$, $\chi^{\delta(S \cup \{j\})}$ and $\chi^{\delta(S \cup \{i,j\})}$ satisfy equality $b.x = 0$. Then, $b_{ij} = 0$ holds.

Lemma 4.2. Let b a vector in $R^{\binom{n}{2}}$. Let i, j, k be distinct elements in $[1, n]$ and S be a subset of $[1, n] - \{i, j, k\}$ such that the cut vectors $\chi^{\delta(S \cup \{j\})}$, $\chi^{\delta(S \cup \{k\})}$, $\chi^{\delta(S \cup \{i,j\})}$ and $\chi^{\delta(S \cup \{i,k\})}$ satisfy equality $b.x = 0$. Then, $b_{ij} = b_{ik}$ holds.

4.1. Proof of Theorem 3.1

We show that the gyroscope inequality is facet inducing for the inequicut cone. We distinguish the cases when n is even or odd since the inequality takes a different form.

4.1.1. The gyroscope facet for n even

We consider the gyroscope inequality $Gyr_{2p}.x \leqslant 0$, defined on the nodes $[1, 2p]$. We first remark that this inequality can be decomposed as a linear combination of triangle inequalities, since this observation helps for checking validity and identifying the roots. We define the following triangle inequalities:

$$A.x = x_{1,4} - x_{1,3} - x_{3,4},$$

$$B.x = x_{2,4} - x_{2,3} - x_{3,4},$$

$$T_i.x = x_{i,i+1} - x_{1,i} - x_{1,i+1}$$

and

$$U_i.x = x_{i,i+1} - x_{2,i} - x_{2,i+1}$$

for all odd i with $5 \leqslant i \leqslant 2p-1$. Recall that any triangle inequality $T.x \leqslant 0$ is valid for C_n and that $T(\delta(S)) = 0, -2$ for any cut $\delta(S)$. One can check easily that

$$Gyr_{2p}.x = A.x - B.x + \sum_{\substack{5 \leqslant i \leqslant 2p-1 \\ i \text{ odd}}} (T_i.x + U_i.x). \tag{7}$$

We first verify the validity for IC_{2p} of the inequality $Gyr_{2p}.x \leqslant 0$. For this, assume that $\delta(S)$ is a cut such that $Gyr_{2p}(\delta(S)) > 0$ with, for instance, $3 \in S$. Then, $A(\delta(S)) = T_i(\delta(S)) = U_i(\delta(S) = 0$ for all i and $B(\delta(S)) = -2$; this implies that $2, 4 \notin S$, $1 \in S$ and $|S \cap \{i, i+1\}| = 1$ for all odd i, $5 \leqslant i \leqslant 2p-1$, i.e. $\delta(S)$ is, in fact, an equicut. Therefore, the inequality $Gyr_{2p}.x \leqslant 0$ is violated only by some equicuts and, hence, it is valid for IC_{2p}.

The description of the roots is obtained by distinguishing the possible values for the triangle inequalities A, B, T_i, U_i composing Gyr_{2p}. Indeed, $Gyr_{2p}.\chi^{\delta(S)} = 0$ if and only if, either $\delta(S)$ defines a root of all the triangle inequalities A, B, T_i, U_i for each i, or $B.\chi^{\delta(S)} = -2$ and $\delta(S)$ defines a root of all the triangle inequalities A, T_i, U_i except one. It is then easy to verify that S is indeed of the types (A1)–(A3) given in Section 3.6.

The identification of the roots for the other facets Gyr_{2p+1}, Dom_{2p}, Dom_{2p+1}, But_{2p} is done in a similar manner, so we will omit the details in the next cases.

We now prove that $Gyr_{2p}.x \leqslant 0$ induces a facet of IC_{2p}. For this, take a valid inequality $b.x \leqslant 0$ for IC_{2p} such that $b.x = 0$ whenever $Gyr_{2p}.x = 0$ for inequicuts x; we verify that $b = \alpha Gyr_{2p}$ for some scalar α through the following statements.

$$\left.\begin{array}{l} \text{For } i, j \text{ with } 5 \leqslant i < j \leqslant 2p \text{ and } j \neq i+1 \text{ if } i \text{ is odd, } b_{ij} = 0 \text{ holds.} \\ \text{For } i \text{ with } 5 \leqslant i \leqslant 2p,\ b_{3i} = b_{4i} = 0 \text{ and } b_{34} = 0. \end{array}\right\} \tag{8}$$

To check statement (8), take i, j as in (8); then the sets $\{i\}$, $\{j\}$, $\{i, j\}$ define roots; hence, taking the empty set as S, Lemma 4.1 implies that $b_{ij} = 0$. For $5 \leqslant i \leqslant 2p$, the sets $\{i\}$, $\{3\}$, $\{4\}$, $\{3, 4\}$, $\{3, i\}$, $\{4, i\}$ all define roots, implying similarly that $b_{3i} = b_{4i} = b_{34} = 0$.

$$\text{For } i \text{ odd with } 5 \leqslant i \leqslant 2p-1,\ b_{1i} = b_{1,i+1} \text{ and } b_{2i} = b_{2,i+1}. \tag{9}$$

For this, let S be the set of all odd j with $5 \leqslant j \leqslant 2p-1$ and $j \neq i$; then, $S \cup \{i\}$, $S \cup \{i+1\}$, $S \cup \{1, i\}$, $S \cup \{1, i+1\}$, $S \cup \{2, i\}$, $S \cup \{2, i+1\}$ define inequicut roots which, from Lemma 4.2, implies that $b_{1i} = b_{1,i+1}$ and $b_{2i} = b_{2,i+1}$.

There exist scalars α, α' such that

$$b_{13} = b_{1i} = \alpha,\ b_{24} = b_{2i} = \alpha' \text{ for all } i, 5 \leqslant i \leqslant 2p. \tag{10}$$

For this, with S as above, apply Lemma 4.2 to the roots defined by the sets $S \cup \{i\}$, $S \cup \{3\}$, $S \cup \{4\}$, $S \cup \{1,i\}$, $S \cup \{3,i\}$, $S \cup \{2,4\}$, $S \cup \{2, i\}$.

$$b_{23} = -\alpha,\ b_{14} = -\alpha'.\ b_{i,i+1} = -\alpha - \alpha' \quad \text{for all odd } i,\ 5 \leqslant i \leqslant 2p-1. \tag{11}$$

This follows from the fact that the sets $\{3\}$, $\{4\}$, $(i\}$ define roots. We conclude the proof with the next statement.

$$\alpha = \alpha' \quad \text{and} \quad b_{12} = 0. \tag{12}$$

Since the sets $S = \{1, 3, 7, 9, \ldots, 2p-1\}$ and $S \cup \{5, 6\}$ both define roots, we have that $0 = b(\delta(S \cup \{5, 6\})) - b(\delta(S))$, implying that $\alpha = \alpha'$. The relation $b_{12} = 0$ follows by using the fact that $\{1, 5, 7, 9, \ldots, 2p-1\}$ defines a root.

4.1.2. The gyroscope facet for n odd

The gyroscope inequality $Gyr_{2p+1}.x \leqslant 0$ is defined on the $2p+1$ nodes $[1, 2p+1]$; if $Gyr_{2p}.x \leqslant 0$ denotes the (even) gyroscope inequality defined on nodes $[1, 2p]$, then, one can verify easily that

$$Gyr_{2p+1}.x = Gyr_{2p}.x + T.x - T'.x, \tag{13}$$

where $T.x = x_{12} - x_{1,2p+1} - x_{2,2p+1}$, $T'.x = x_{2p-1,2p} - x_{2p-1,2p+1} - x_{2p,2p+1}$ are two triangle inequalities; therefore, $Gyr_{2p+1}.x$ can also be decomposed as a linear combination of triangle inequalities.

Let $\delta(S)$ be a cut such that $Gyr_{2p+1}(\delta(S)) > 0$ and $S \subset [1, 2p]$. Then, either $Gyr_{2p}(\delta(S)) > 0$ implying that $|S| = p$, and thus $\delta(S)$ is an equicut, or $Gyr_{2p}(\delta(S)) = T(\delta(S)) = 0$, $T'(\delta(S)) = -2$, implying easily that $|S| = p+1$, and thus $\delta(S)$ is an equicut. Hence, $Gyr_{2p+1}.x \leqslant 0$ is violated only by equicuts, i.e. is valid for IC_{2p+1}.

We show that the inequality $Gyr_{2p+1}.x \leqslant 0$ defines a facet of IC_{2p+1}; let $b.x \leqslant 0$ be a valid inequality for IC_{2p+1} such that $b.x = 0$ whenever $Gyr_{2p+1}.x = 0$ for all inequicuts x. We show that $b = \alpha Gyr_{2p+1}$ for some positive scalar α.

$$b_{ij} = 0 \text{ for each } (i,j) \text{ which is not an edge of the supporting graph of } Gyr_{2p+1}. \tag{14}$$

This statement follows from Lemma 4.1 applied to the roots defined by the sets $\{i\}$, $\{j\}$, $\{i,j\}$, where either $4 \leqslant i < j \leqslant 2p$ with $j \neq i+1$ if i is odd, or $4 \leqslant i \leqslant 2p-2$ and $j = 2p+1$, or $i = 3$ and $4 \leqslant j \leqslant 2p+1$.

The proof of the next statement (15) is analogous to that of statements (9)–(11), above and hence is omitted.

$$\left.\begin{array}{l}\text{There exist scalars } \alpha,\alpha' \text{ such that } b_{13}=b_{1i}=b_{1,i+1}=\alpha,\\ b_{24}=b_{2i}=b_{2,i+1}=\alpha' \text{ for odd } i,\ 5\leqslant i\leqslant 2p-1, \text{ and } b_{23}=-\alpha,\\ b_{14}=-\alpha',\ b_{i,i+1}=-\alpha-\alpha' \text{ for odd } i,\ 5\leqslant i\leqslant 2p-3.\end{array}\right\} \tag{15}$$

$$\alpha=\alpha'. \tag{16}$$

For this, observe that both sets $S=\{1,3,7,9,\dots,2p-1\}$ and $S\cup\{5,6,2p+1\}$ define roots, implying:

(∗) $\quad 0=b(\delta(S\cup\{5,6,2p+1\}))-b(\delta(S))=b_{1,2p+1}-b_{2,2p+1}+2(\alpha-\alpha')$

Then, if $S'=\{3,4,5,7,\dots,2p-1,2p+1\}$, both $S'\cup\{1\}$, $S'\cup\{2\}$ define roots, implying: $0=b(\delta(S'\cup\{1\}))-b(\delta(S'\cup\{2\}))$ and thus

(∗∗) $\quad b_{1,2p+1}=b_{2,2p+1}$.

Then, (∗) and (∗∗) yield $\alpha=\alpha'$.

$$b_{1,2p+1}=b_{2,2p+1}=\alpha \quad\text{and}\quad b_{2p,2p+1}=b_{2p-1,2p+1}=b_{2p-1,2p}=-\alpha. \tag{17}$$

Since $0=b(\delta(2p-1))=2\alpha+b_{2p-1,2p}+b_{2p-1,2p+1}$ and $0=b(\delta(2p))=2\alpha+b_{2p-1,2p}+b_{2p,2p+1}$, we deduce that $b_{2p-1,2p+1}=b_{2p,2p+1}$. Also, $0=b(\delta(2p+1))=2b_{1,2p+1}+2b_{2p,2p+1}$, and thus $b_{2p,2p+1}=-b_{1,2p+1}$. Finally, for $S=\{5,7,\dots,2p-3\}$, the sets $S\cup\{3\}$, $S\cup\{2p+1\}$, $S\cup\{1,3\}$ and $S\cup\{1,2p+1\}$ define roots which, from Lemma 4.2, implies that $b_{1,2p+1}=b_{1,3}=\alpha$, stating assertion (17).

We conclude the proof by verifying that $b_{12}=-\alpha$ using the fact that $\{1,5,7,\dots,2p-1\}$ defines a root.

4.2. *Proof of Theorem 3.2*

We prove that the domino inequality is facet inducing for the inequicut cone, again we distinguish the cases n even or odd.

4.2.1. *The domino facet for n even*

We observe first that the domino inequality $Dom_{2p}.x\leqslant 0$, defined on nodes [1, 2p], can be written as follows:

$$Dom_{2p}.x=2x_{12}+\sum_{\substack{1\leqslant i\leqslant 2p-3\\ i\,\mathrm{odd}}}(A_i.x+B_i.x), \tag{18}$$

where $A_i.x=x_{i+2,i+3}-x_{i,i+2}-x_{i,i+3}$ and $B_i.x=x_{i+2,i+3}-x_{i+1,i+2}-x_{i+1,i+3}$.

Obviously, the only cuts violating inequality $Dom_{2p}.x\leqslant 0$ are the equicuts $\delta(S)$ for which $|S\cap\{i,i+1\}|=1$ for all odd i; therefore, the domino inequality is valid over IC_{2p}.

We prove that the domino inequality $Dom_{2p}.x \leqslant 0$ defines a facet of IC_{2p}. For this, take a valid inequality $b.x. \leqslant 0$ for IC_{2p} such that $b.x=0$ whenever $Dom_{2p}.x=0$. We verify below that $b=\alpha Dom_{2p}$ for some scalar α. The next lemma can easily be verified.

Lemma 4.3. *For $s=3$ or 4, let $p_1, p_2, \ldots, p_s$ be s distinct nodes in $[1, 2p]$ and S be a subset of $[1, 2p]-\{p_1, p_2, \ldots, p_s\}$. If the cuts $\delta(S)$, $\delta(S \cup \{p_i\})$, for $1 \leqslant i \leqslant s$, and $\delta(S \cup \{p_1, \ldots, p_s\})$ satisfy the equality $b.x=0$, then $\sum_{1 \leqslant t < t' \leqslant s} b_{p_t, p_{t'}}=0$ holds.*

$$\left.\begin{array}{l} b_{12}+b_{13}+b_{23}=b_{12}+b_{14}+b_{24}=b_{13}+b_{14}+b_{34}=b_{23}+b_{24}+b_{34}=0 \\ \text{and, for } i \text{ odd with } 1 \leqslant i \leqslant 2p-3, \\ b_{i,i+1}+b_{i,i+2}+b_{i,i+3}+b_{i+1,i+2}+b_{i+1,i+3}+b_{i+2,i+3}=0. \end{array}\right\} \quad (19)$$

The relation $b_{13}+b_{12}+b_{23}=0$ follows by applying Lemma 4.3 with $s=3$, $S=\{5, 7, \ldots, 2p-1\}$ and the three nodes $1, 2, 3$; the other relations of the first statement of (19) can be obtained similarly. The second set of relations of (19) follows by applying Lemma 4.3 with $s=4$, $S=\{1, 3, \ldots, i-2\} \cup \{i+4, i+6, \ldots, 2p-1\}$ and the four nodes $i, i+1, i+2, i+3$ with i odd, $1 \leqslant i \leqslant 2p-3$.

$$\text{For some scalar } \alpha, \quad b_{13}=b_{23}=b_{14}=b_{24}=\alpha \quad \text{and} \quad b_{12}=b_{34}=-2\alpha. \quad (20)$$

Applying Lemma 4.2 to the roots defined by $\{1\}, \{2\}, \{1, 3\}, \{2, 3\}$ yields $b_{13}=b_{23}$; by symmetry, we obtain that $b_{13}=b_{23}=b_{14}=b_{24}=\alpha$. Since $\delta(\{1\})$ is a root, we obtain that $b_{12}=-2\alpha$ and, similarly, $b_{34}=-2\alpha$.

$$\left.\begin{array}{l} b_{i,i+2}=b_{i,i+3}=b_{i+1,i+2}=b_{i+1,i+3}=\alpha \quad \text{and} \quad b_{i,i+1}=-2\alpha \\ \text{for odd } i, \ 3 \leqslant i \leqslant 2p-3. \end{array}\right\} \quad (21)$$

Take i odd with $3 \leqslant i \leqslant 2p-3$. For $S=\{i+4, i+6, \ldots, 2p-1\}$, the sets $S \cup \{i+2\}$, $S \cup \{i+3\}$, $S \cup \{i, i+2\}$, $S \cup \{i, i+3\}$ define roots which, from Lemma 4.2, implies that $b_{i,i+2}=b_{i,i+3}$; similarly, $b_{i+1,i+2}=b_{i+1,i+3}$. For $S'=\{1, 3, \ldots, i-2\}$, the sets $S' \cup \{i\}$, $S' \cup \{i+1\}$, $S' \cup \{i, i+2\}$, $S' \cup \{i+1, i+2\}$ define roots, implying that $b_{i,i+2}=b_{i+1,i+2}$; similarly, $b_{i,i+3}=b_{i+1,i+3}$. For $S''=S \cup S'$, the sets $S'' \cup \{i-2\}$, $S'' \cup \{i+2\}$, $S'' \cup \{i, i-2\}$, $S'' \cup \{i, i+2\}$ define roots, implying that $b_{i,i-2}=b_{i,i+2}$. One then deduces (21) by induction on i using (19) and (20).

$$\left.\begin{array}{l} b_{ij}=0 \text{ for all } (i,j) \text{ which is not an edge of the supporting} \\ \text{graph of } Dom_{2p}. \end{array}\right\} \quad (22)$$

Take i, k odd such that $1 \leqslant i \leqslant 2p-5$, $i+4 \leqslant k \leqslant 2p-1$ and $S=\{1, 3, \ldots, i-2\} \cup \{k+2, \ldots, 2p-1\}$; then the sets S, $S \cup \{i\}$, $S \cup \{k\}$, $S \cup \{i, k\}$ define roots which, from Lemma 4.1, implies that $b_{ik}=0$; the result follows then by symmetry.

4.2.2. The domino facet for n odd

The domino inequality $Dom_{2p+1}.x \leqslant 0$ is defined on nodes $[1, 2p+1]$ and, if $Dom_{2p}.x \leqslant 0$ is defined on nodes $[1, 2p]$, then, one verifies easily that

$$Dom_{2p+1}.x = Dom_{2p}.x + T.x - T'.x, \quad (23)$$

where

$$T.x=x_{2p-3,2p-2}-x_{2p-3,2p+1}-x_{2p-2,2p+1} \text{ and}$$

$$T'.x=x_{2p-1,2p}-x_{2p-1,2p+1}-x_{2p,2p+1}.$$

In order to verify validity for IC_{2p+1}, take a cut $\delta(S)$ such that $Dom_{2p+1}(\delta(S))>0$ and $S\subseteq[1,2p]$. Either, $Dom_{2p}(\delta(S))>0$, or $Dom_{2p}(\delta(S))=T(\delta(S))=0$ and $T'(\delta(S))=-2$ implying in both cases that $\delta(S)$ is an equicut.

Take a valid inequality $b.x\leqslant 0$ for IC_{2p+1} such that $b.x=0$ whenever $Dom_{2p+1}.x=0$; we show that $b=\alpha Dom_{2p+1}$ for some scalar α. The proof of the next statement is similar to that of assertions (19)–(21) and thus omitted.

$$\left.\begin{array}{l} b_{12}+b_{13}+b_{23}=b_{23}+b_{24}+b_{34}=0 \text{ and, for odd } i \text{ with } 1\leqslant i\leqslant 2p-7 \\ \text{or } i=2p-3,\ b_{i,i+1}+b_{i,i+2}+b_{i,i+3}+b_{i+1,i+2}+b_{i+1,i+3}+b_{i+2,i+3}=0. \\ \text{There exists a scalar } \alpha \text{ such that } b_{i,i+2}=b_{i,i+3}=b_{i+1,i+2} \\ =b_{i+1,i+3}=\alpha \text{ and, therefore, } b_{12}=b_{34}=\cdots=b_{2p-5,2p-4}=-2\alpha \text{ and} \\ b_{2p-3,2p-2}+b_{2p-1,2p}=-4\alpha. \end{array}\right\} \quad (24)$$

$$\left.\begin{array}{l} b_{ij}=0 \text{ for all } (i,j) \text{ which is not an edge of the supporting} \\ \text{graph of } Dom_{2p+1} \end{array}\right\} \quad (25)$$

If $1\leqslant i<j\leqslant 2p$, $b_{ij}=0$ is obtained in a similar way as in (22) and $b_{i,2p+1}=0$ for $i\in[1,2p-4]$ follows from Lemma 4.1 applied to roots $\delta(S)$, $\delta(S\cup\{i\})$, $\delta(S\cup\{2p+1\})$, $\delta(S\cup\{i,2p+1\})$, where $S=\{1,3,\ldots,i-2\}$.

$$\left.\begin{array}{l} b_{2p-3,2p-2}=-3\alpha, b_{2p-1,2p}=-\alpha, b_{2p+1,2p-1}=b_{2p+1,2p}=-\alpha \text{ and} \\ b_{2p+1,2p-3}=b_{2p+1,2p-2}=\alpha. \end{array}\right\} \quad (26)$$

From Lemma 4.2 applied to roots $\delta(\{2p-1,2p-3\})$, $\delta(\{2p-1,2p-2\})$, $\delta(\{2p-1,2p-3,2p+1\})$, $\delta(\{2p-1,2p-2,2p+1\})$, we obtain that $b_{2p+1,2p-3}=b_{2p+1,2p-2}$; similarly, $b_{2p+1,2p-1}=b_{2p+1,2p}$. Using root $\delta(2p+1)$, we obtain that $0=b(\delta(2p+1))$ and thus $b_{2p-1,2p}+b_{2p-1,2p+1}=-2\alpha$. Finally, using root $\delta(\{2p-3,2p-1,2p\})$, we obtain that $0=b_{2p-3,2p-2}+b_{2p,2p+1}+4\alpha$ which, together with relation $b_{2p-3,2p-2}+b_{2p-1,2p}=-4\alpha$ from (24), implies that $b_{2p,2p+1}=b_{2p,2p-1}=-\alpha$. This concludes the proof.

4.3. Proof of Theorem 3.3

The butterfly inequality $But_{2p}.x\leqslant 0$ is defined on nodes $[1,2p]$ and one can check that

$$But_{2p}.x=\sum_{1\leqslant i\leqslant p-1} T_i.x+\sum_{p+1\leqslant i\leqslant 2p-2} U_i.x-T_0.x, \quad (27)$$

where

$$T_0.x = x_{2p-1,2p} - x_{p,2p-1} - x_{p,2p},$$

$$T_i.x = x_{i,p+1} - x_{ip} - x_{p,p+1} \quad \text{for } 1 \leqslant i \leqslant p-1$$

and

$$U_i.x = x_{2p-1,2p} - x_{i,2p-1} - x_{i,2p} \quad \text{for } p+1 \leqslant i \leqslant 2p-2.$$

One checks easily that the only cut violating inequality $But_{2p}.x \leqslant 0$ is the equicut $\delta([1,p])$.

Take a valid inequality $b.x \leqslant 0$ for IC_{2p} such that $b.x = 0$ whenever $But_{2p}.x = 0$; we show that $b = \alpha But_{2p}$ for some scalar α, thus stating that the butterfly inequality is facet inducing for IC_{2p}. The cases $p = 3, 4$ were checked by computer, so we can assume that $p \geqslant 5$.

$$\left.\begin{array}{l} b_{ij}=0 \text{ for all } (i,j) \text{ which is not an edge of the supporting} \\ \text{graph of } But_{2p}. \end{array}\right\} \tag{28}$$

This fact follows by applying Lemma 4.1 successively to:
— roots $\delta(i), \delta(j), \delta(\{i,j\})$ with $1 \leqslant i < j \leqslant p-1$,
— roots $\delta(S), \delta(S \cup \{i\}), \delta(S \cup \{j\}), \delta(S \cup \{i,j\})$ for $S = \{2p-1\}$ and $p+2 \leqslant i < j \leqslant 2p-2$ or $1 \leqslant i \leqslant p-1$ and $p+2 \leqslant j \leqslant 2p-2$,
— roots $\delta(S), \delta(S \cup \{i\}), \delta(S \cup \{2p\}), \delta(S \cup \{i, 2p\})$ for $S = \{p, p+1\}$ and $1 \leqslant i \leqslant p$-1, and similarly, exchanging $2p$ with $2p-1$,
— roots $\delta(S),. \delta(S \cup \{i\}), \delta(S \cup \{p, i\}), \delta(S \cup \{p\})$ for $S = \{p+1, 2p\}$ and $p+2 \leqslant i \leqslant 2p-2$,
— roots $\delta(S), \delta(S \cup \{i\}), \delta(S \cup \{p+1\}), \delta(S \cup \{p+1, i\})$ for $S = \{2p-1\}$ and $p+1 \leqslant i \leqslant 2p-2$.

$$\text{For some scalar } \alpha, \quad b_{pi} = \alpha, \quad b_{p+1,i} = -\alpha \quad \text{for } 1 \leqslant i \leqslant p-1. \tag{29}$$

If $S = \{p+1, 2p\}$, $1 \leqslant i < j \leqslant p-1$, then $S \cup \{i\}, S \cup \{j\}, S \cup \{p, i\}, S \cup \{p, j\})$ define roots, implying that $b_{pi} = b_{pj}$; we set $\alpha = b_{pi}$. Since $\{i\}$ defines a root, one has $0 = b.\delta(i) = b_{pi} + b_{p+1,i}$, implying that $b_{p+1,i} = -\alpha$.

$$b_{2p-1,i} = \beta, \qquad b_{2p,i} = \gamma \quad \text{for } p+2 \leqslant i \leqslant 2p-2. \tag{30}$$

Apply Lemma 4.2 to the roots defined by the sets $S \cup \{i\}, S \cup \{j\}, S \cup \{2p, i\}, S \cup \{2p, j\}$ with $S = [1, p]$, $p+2 \leqslant i < j \leqslant 2p-2$ and similarly replacing $2p$ by $2p-1$.

Using the roots defined by the sets $\{p+1, 2p\}$, $\{p+1, 2p-1\}$, $\{p, p+1, 2p\}$, $\{p, p+1, 2p-1\}$, we deduce, respectively, the following relations:

$$0 = -\alpha(p-1) + b_{p,p+1} + b_{p+1,2p-1} + b_{p,2p} + \gamma(p-3) + b_{2p,2p-1},$$

$$0 = -\alpha(p-1) + b_{p,p+1} + b_{p+1,2p} + b_{p,2p-1} + \beta(p-3) + b_{2p,2p-1},$$

$$0 = b_{p,2p-1} + b_{p+1,2p-1} + b_{2p,2p-1} + \gamma(p-3),$$

$$0 = b_{p,2p} + b_{p+1,2p} + b_{2p,2p-1} + \beta(p-3),$$

yielding

$$b_{p,2p}=b_{p,2p-1} \tag{31}$$

Using the roots defined by $\{p,p+1,p+2,2p\}$ and $\{p,p+1,p+2,2p-1\}$, we obtain the following relations:

$$0=b_{p,2p-1}+b_{p+1,2p-1}+b_{2p-1,2p}+\gamma(p-4)+\beta$$

$$0=b_{p,2p}+b_{p+1,2p}+b_{2p-1,2p}+\beta(p-4)+\gamma$$

which, together with (31) and above relations, implies

$$\beta=\gamma \text{ and } b_{p+1,2p-1}=b_{p+1,2p}. \tag{32}$$

From root $\delta(\{p,p+1\})$, we obtain $0=2b_{p,2p}+2b_{p+1,2p}$, yielding

$$b_{p,2p}=-b_{p+1,2p}. \tag{33}$$

From root $\delta(\{2p-1\})$, we obtain $0=b_{2p-1,p}+b_{2p-1,p+1}+b_{2p-1,2p}+\beta(p-3)$ and thus

$$b_{2p-1,2p}=-\beta(p-3). \tag{34}$$

From root $\delta([p+2,2p-2]\cup\{p+1,2p-1\})$, we obtain

$$b_{p,p+1}=\alpha(p-1). \tag{35}$$

From root $\delta([1,p]\cup\{p+2\})$, we obtain

$$b_{p,2p}=-\beta. \tag{36}$$

For finishing the proof, it remains only to check that $\alpha=\beta$, which follows using root $\delta([2,p])$.

Note added in proof. Recently, J. Kahn and G. Kalai disproved Borsuk's conjecture, using the set of equicuts of K_n, for $n=4k$, k prime power.

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