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A Few Applications of Negative-Type Inequalities

M. Deza[1] and H. Maehara[2]

[1] CNRS, Paris, France
[2] Ryukyu University, Okinawa, Japan

Abstract. Applying the negative-type inequalities for square Euclidean distance, we present (1) a parallelotope theorem (a generalization of the parallelogram theorem), (2) a short proof of Rankin's theorem for the maximum number of dispersed points on a sphere, and (3) a proof of impossibility of a certain geometric embedding for some graphs.

1. Introduction

Let $X = \{p_1, \ldots, p_N\}$ be a finite metric space. It is well known that X is isometrically embeddable in Euclidean space if and only if the $N \times N$ matrix (D_{ij}) is of *negative-type* (where $D_{ij} = d(p_i, p_j)^2$), that is, for every real vector $v = (v_1, \ldots, v_N)$,

$$\sum_i v_i = 0 \Rightarrow \sum_{i,j} D_{ij} v_i v_j \leq 0,$$

see, e.g. Schoenberg [7] or Seidel [8]. This condition is equivalent to the following system of inequalities (see e.g. Assouad-Deza [1]): For any $n > 0$ and $2n$ sequence of (not necessarily distinct) points $x_1, \ldots, x_n, y_1, \ldots, y_n$ of X,

$$\sum_{i<j} d(x_i, x_j)^2 + \sum_{i<j} d(y_i, y_j)^2 \leq \sum_{i,j} d(x_i, y_j)^2. \qquad (*)$$

Thus, if $x_1, \ldots, x_n, y_1, \ldots, y_n$ are points in a Euclidean space, then the inequality $(*)$ always holds. The inequality $(*)$ is called the *negative-type inequality* for square distance.

In this paper, applying the negative-type inequality, we prove the parallelotope theorem (Theorem 3) which is a generalization of the parallelogram theorem. We also present a short proof of Rankin's theorem on the maximum size of dispersed set on a sphere of radius $< 2^{-1/2}$, and a short proof of impossibility of (α, β)-embedding for complete bipartite graph $K_{n,n}$ of large order.

Throughout this paper, a *point* means a point in a Euclidean space, and a *point-set* means a subset of a Euclidean space.

2. Inequalities

First we show the inequality (∗) in a form of equality.

Theorem 1. *Let* $X = \{x_1, \ldots, x_n\}$, $Y = \{y_1, \ldots, y_n\}$ *be two point-sets in a Euclidean space. Then*

$$\sum_{i,j} d(x_i, y_j)^2 = \sum_{i<j} d(x_i, x_j)^2 + \sum_{i<j} d(y_i, y_j)^2 + n^2 d(p,q)^2,$$

where p *and* q *are barycenters of* X *and* Y, *that is,*

$$p = (x_1 + \cdots + x_n)/n, \; q = (y_1 + \cdots + y_n)/n.$$

Remark. Since the bove equality holds for any point-sets X and Y of size n, it also holds for *multi-sets* X and Y.

Proof. Using the inner product,

$$\begin{aligned} d(x_i, y_j)^2 &= (x_i - y_j, x_i - y_j) = (x_i - p + p - q + q - y_j, x_i - p + p - q + q - y_j) \\ &= d(x_i, p)^2 + d(p,q)^2 + d(q, y_j)^2 + 2(x_i - p, p - q) + 2(x_i - p, q - y_j) \\ &\quad + 2(p - q, q - y_j). \end{aligned}$$

Summing up for $i = 1, \ldots, n$ and $j = 1, \ldots, n$, we have

$$\sum_{i,j} d(x_i, y_j)^2 = n \sum_i d(x_i, p)^2 + n \sum_j d(q, y_j)^2 + n^2 d(p,q)^2.$$

Now, since

$$\begin{aligned} 2 \sum_{i<j} d(x_i, x_j)^2 &= \sum_{i,j} d(x_i, x_j)^2 \\ &= \sum_{i,j} (x_i - x_j, x_i) - \sum_{i,j} (x_i - x_j, x_j) \\ &= n \sum_i (x_i - p, x_i) + n \sum_j (x_j - p, x_j) = 2n \sum_i (x_i - p, x_i) \\ &= 2n \sum_i (x_i - p, x_i - p) = 2n \sum_i d(x_i, p)^2, \end{aligned}$$

we have

$$n \sum_i d(x_i, p)^2 = \sum_{i<j} d(x_i, x_j)^2.$$

Similarly,

$$n \sum_j d(q, y_j)^2 = \sum_{i<j} d(y_i, y_j)^2.$$

Hence the theorem follows. □

In the following, a *graph* implies a simple graph with vertices in a Euclidean space, whose edges are line segments connecting two vertices. The complete graph with vertex set X is denoted by $K(X)$, and the complete m-partite graph with partite sets $X_1, \ldots, X_m$ is denoted by $K(X_1, \ldots, X_m)$.

For a graph G with edge set E, we denote by ssG the sum of the squared edge-lengths, that is,

$$ssG = \sum_{xy \in E} d(x,y)^2.$$

Using these notations, Theorem 1 can be stated as follows:

For any two point-sets X and Y of the same size n,

$$ssK(X,Y) \geq ssK(X) + ssK(Y)$$

holds, and the equality holds if and only if the barycenters of X and Y coincide.

Theorem 2. *Let $X_1, \ldots, X_m$ be m point-sets of the same size n. Then*

$$ssK(X_1, \ldots, X_m) \geq (m-1)\{ssK(X_1) + \cdots + ssK(X_m)\}$$

and the equality holds if and only if the barycenters of $X_1, \ldots, X_m$ coincide.

Proof. Theorem 1 implies that for $i \neq j$,

$$ssK(X_i, X_j) \geq ssK(X_i) + ssK(X_j),$$

in which equality holds if and only if the barycenters of X_i and X_j coincide. Since

$$ssK(X_1, \ldots, X_m) = \sum_{i<j} ssK(X_i, X_j),$$

we have

$$ssK(X_1, \ldots, X_m) \geq \sum_{i<j} \{ssK(X_i) + ssK(X_j)\}$$
$$= (m-1)\{ssK(X_1) + \cdots + ssK(X_m)\},$$

and the equality holds if and only if the barrycenters of X_is coincide. □

Corollary 1. *For a point-set $X = \{x_1, \ldots, x_n, y_1, \ldots, y_n\}$, we always have*

$$ssK(X) \geq n \sum_i d(x_i, y_i)^2,$$

where the equality holds if and only if the n midpoints of the n line segments $x_i y_i$ coincide.

Proof. Let $X_i = \{x_i, y_i\}$, $i = 1, \ldots, n$. Then by Theorem 2,

$$ssK(X_1, \ldots, X_n) \geq (n-1) \sum_i d(x_i, y_i)^2.$$

Adding $\sum_i d(x_i, y_i)^2$ to both sides, we have the corollary. □

3. The Parallelotope Theorem

Let x, x', y, y' be four distinct points. Then the midpoints of the two line segments xx' and yy' coincide if and only if $xyx'y'$ forms a (possibly degenerate) parallelogram. Hence, by the above corollary, we have the well known *parallelogram theorem*:

For any four points x, x', y, y',

$$d(x,y)^2 + d(y,x')^2 + d(x',y')^2 + d(y',x)^2 \geq d(x,x')^2 + d(y,y')^2.$$

The equality holds if and only $xyx'y'$ is a parallelogram.

This theorem can be generalized to higher dimension.

For a convex polytope P, $G(P)$ denotes its graph, i.e. the 1-skeleton of P. Hence $ssG(P)$ is the sum of the squared edges of P. A line segment connecting two vertices of a polytope P is called a *diagonal* of P if it is not contained in a proper face of P. For example, a three dimensional box has exactly four diagonals, and an n-dimensional cube has 2^{n-1} diagonals. Two polytopes P and Q are said to be *combinatorially equivalent* if there is a one to one correspondence between the set of faces of P and the set of faces of Q which preserves incidence-relation.

An *n-parallelotope* is the vector-sum of n segments with a common endpoint, such that none is contained in the affine hull of the others. Thus 2-parallelotope is a parallelogram, and 3-parallelotope is a parallelepiped. An n-parallelotope is combinatorially equivalent to an n-cube.

Theorem 3. *Let P_n be an n-polytope, $n \geq 2$, which is combinatorially equivalent to an n-dimensional cube. Then*

$$ssG(P_n) \geq \textit{the sum of squared diagonals of } P_n,$$

where equality holds if and only if P_n is an n-parallelotope.

Proof. The proof is by induction on n. If $n = 2$, then the theorem is the parallelogram theorem. Suppose that the theorem is true up to $n - 1$, and consider the case n.

Let us denote by $D(F)$ the sum of the squared diagonals of a face F of P_n, and by D_k the sum of $D(F)$ for all k-faces F of P_n. If $k < n$, then since a k-face F is combinatorially equivalent to a k-cube, it follows from the inductive hypothesis that

$$ssG(F) \geq D(F). \tag{1}$$

The number f_k of k-faces of P_n is

$$f_k = 2^{n-k}\binom{n}{k},$$

see e.g. Grumbaum [3, p. 57]. Hence, P_n has $f_1 = n2^{n-1}$ edges. Since a k-face F has $k2^{k-1}$ edges, summing the inequality (1) for all k-faces F of P_n, we have

$$(f_k k2^{k-1}/f_1)ssG(P_n) \geq D_k.$$

Since

$$f_k k2^{k-1}/f_1 = \binom{n-1}{k-1}$$

we have

$$\binom{n-1}{k-1} ssG(P_n) \geq D_k.$$

Summing up from $k = 2$ to $k = n - 1$, we have

$$(2^{n-1} - 2)ssG(P_n) \geq D_2 + \cdots + D_{n-1}.$$

Adding $ssG(P_n)$ to both sides, we have

$$(2^{n-1} - 1)ssG(P_n) \geq ssG(P_n) + D_2 + \cdots + D_{n-1}.$$

Now, since

$$ssG(P_n) + D_2 + \cdots + D_{n-1} + D_n \geq 2^{n-1}D_n \tag{2}$$

by Corollary 1,

$$(2^{n-1} - 1)ssG(P_n) \geq (2^{n-1} - 1)D_n,$$

and hence we have

$$ssG(P_n) \geq D_n. \tag{3}$$

The equality holds if and only if the equalities in (1) and (2) hold simultaneously, i.e., all proper faces of P_n are parallelotopes and the diagonals of P_n bisect each other. Thus the equality in (3) holds if and only if P_n is an n-parallelotope. □

Remark. Let G_n be a graph in a Euclidean space which is isomorphic to th 1-skeleton of an n-cube. Then the proof of the above theorem shows that

$$ssG_n \geq D_n,$$

where D_n denotes the sum of $d(x, y)^2$ for those pairs of vertices x, y with mutual 'graph-distance' n (that is, the minimum number of edges in a path from x to y is n). Furthermore, the equality holds if and only if G_n is the graph of a (possibly degenerate) n-parallelotope.

Corollary 2. *For any n-parallelotope of unit edge-length, the sum of the squared diagonals is equal to* $n2^{n-1}$. □

4. Circumradii and Dispersed Point Set

The *circumsphere* of a nonempty finite set X in a Euclidean space is the sphere of minimum radius that encloses the set X. Its center and radius are called the *circumcenter* and the *circumradius* of X.

Theorem 4. *For any point-set* $X = \{x_1, \ldots, x_n\}$, *the circumradius* R *of* X *satisfies that*

$$n^2R^2 \geq \sum_{i<j} d(x_i, x_j)^2$$

The equality holds if and only if (i) *all points of* X *lie on the surface of the circumsphere and* (ii) *the circumcenter coincides with the barycenter* p *of* X.

Proof. Let q be the circumcenter of X, and let $Y = \{y_1, \ldots, y_n\}$ be the multi-set such that $y_1 = \cdots = y_n = q$. Then applying Theorem 1, we have

$$\begin{aligned} n^2R^2 &\geq \sum_{i,j} d(x_i, y_j)^2 \\ &= \sum_{i<j} d(x_i, x_j)^2 + 0 + n^2 d(p, q)^2. \end{aligned}$$ □

As an application of Theorem 4, we see that the circumradius R of an $(n-1)$-dimensional regular simplex of unit side is given by

$$R = ((n-1)/(2n))^{1/2}.$$

A point-set X is called *dispersed* if $d(x, y) \geq 1$ for every pair of distinct points $x, y \in X$.

The next theorem was proved by Rankin [6]. We present here a very short proof by applying Theorem 4.

Theorem 5. *The maximum size N_R of a dispersed point-set in a closed ball of radius $R < 2^{-1/2}$ in sufficiently high dimensional Euclidean space is given by*

$$N_R = [(1 - 2R^2)^{-1}],$$

where [ ] *denotes the integer part.*

Proof. Let $\{x_1, \ldots, x_n\}$ be a dispersed point-set in a ball of radius R. Then by Theorem 4,

$$n^2R^2 \geq \sum_{i<j} d(x_i, x_j)^2 \geq n(n-1)/2.$$

Hence $n \leq 1/(1 - 2R^2)$, and hence $N_R \leq [(1 - 2R^2)^{-1}]$.

On the other hand, the circumradius of an $(N_R - 1)$-dimensional regular simplex of unit side is

$$((N_R - 1)/(2N_R))^{1/2}$$

Since $N_R \leq 1/(1 - 2R^2)$ implies that $R \geq ((N_R - 1)/(2N_R))^{1/2}$, a ball of radius R contains an $(N_R - 1)$-dimensional regular simplex of unit side. Hence a ball of radius R contains a dispersed point-set of size N_R. □

5. On Geometric Representations of Graphs

An (α, β)-graph is a graph with vertices in a Euclidean space, in which each edge has length at most α and nonadjacent vertices are at least distance β apart. It is known [4] that for any $\alpha > 0$ and for any graph G, there is an (α, α)-graph isomorphic to G. However, for $0 < \alpha < \beta$, there always exists a graph which is not isomorphic to any (α, β)-graph, see Maehara [5]. Here we present a very short proof of this result by applying Theorem 1.

Theorem 6. *If $0 < \alpha < \beta$ and $n > \beta^2/(\beta^2 - \alpha^2)$, then no (α, β)-graph is isomorphic to the complete bipartite graph $K_{n,n}$.*

Proof. Suppose that $K_{n,n}$ is isomorphic to an (α, β)-graph $K(X, Y)$. Then by the definition of (α, β)-graph, we have

$$ssK(X, Y) \leq n^2\alpha^2$$

and

$$\beta^2 n(n-1)/2 \leq ssK(X), \qquad \beta^2 n(n-1)/2 \leq ssK(Y)$$

Since

$$ssK(X) + ssK(Y) \leq ssK(X, Y)$$

by Theorem 1, we have

$$\beta^2 n(n-1) \leq n^2\alpha^2$$

and hence $n \leq \beta^2/(\beta^2 - \alpha^2)$. Therefore, if $n > \beta^2/(\beta^2 - \alpha^2)$ then there is no (α, β)-graph isomorphic to $K_{n,n}$. □

A similar result can be proved for representation by a unit-distance graph. A *unit-distance graph* is a graph with vertices in a Euclidean space, in which two vertices are connected by an edge if and only if there are exactly unit distance apart from each other. It is also known [4] that for any graph G, there is a unit-distance graph isomorphic to G.

In a unit-distance graph, the minimum value of $|d(x, y) - 1|$ for non-adjacent pair of vertices x, y is called the *gap* of the unit distance graph.

Theorem 7. *For any $\varepsilon > 0$, there is a graph which is never isomorphic to a unit-distance graph with gap $> \varepsilon$.*

To prove this theorem, we use the edge-version of *Induced Ramsey Theorem* [2]. It asserts that for any graph G, there is a graph H such that if the edges of H are colored by two colors, then there always appears a monochromatic induced subgraph isomorphic to G. Such a graph H is called an *induced Ramsey graph* for G.

Proof. Let G be the disjoint union of the complete bipartite graph $K_{n,n}$ and its complement $\bar{K}_{n,n}$, where

$$n > (1 + \varepsilon)^2/((1 + \varepsilon)^2 - 1).$$

Let H be an induced Ramsey graph for G. We are going to show that the complement $\bar{H}$ of H is a desired graph.

Suppose that there is a unit-distance graph with gap $> \varepsilon$ which is isomorphic to $\bar{H}$, say, $\bar{H}$ itself is a unit-distance graph with gap $> \varepsilon$. Then, in the graph H, each edge has length either less than $1 - \varepsilon$ or greater than $1 + \varepsilon$, and two non-adjacent vertices are unit distance apart. Now color the edges of length $< 1 - \varepsilon$ by red, and the edges of length $> 1 + \varepsilon$ by blue. Then, since H is an induced Ramsey graph for G, there appears a monochromatic induced subgraph G' of H which is isomorphic to G. If the color of G' is red, then G' is a $(1 - \varepsilon, 1)$-graph. Therefore the induced subgraph $K_{n,n}$ of G is isomorphic to a $(1 - \varepsilon, 1)$-graph. This is impossible by

Theorem 6 because

$$n > (1+\varepsilon)^2/((1+\varepsilon)^2 - 1) > 1/(1-(1-\varepsilon)^2).$$

If the color of G' is blue, then its complement is a $(1, 1+\varepsilon)$-graph. Hence $\bar{G}$ is isomorphic to a $(1, 1+\varepsilon)$-graph. Then, since $\bar{G}$ contains $K_{n,n}$ as an induced subgraph, $K_{n,n}$ is also isomorphic to a $(1, 1+\varepsilon)$-graph. This is impossible because

$$n > (1+\varepsilon)^2/((1+\varepsilon)^2 - 1). \qquad \square$$

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Note added in proof. Theorem 1 is proved in more general setting in: Alexander, R., "Metric averaging in Euclidean and Hilbert spaces", Pacific J. Math. 85 (1979) 1–9.