Annals of Discrete Mathematics 12 (1982) 87–94

ON MAXIMAL EQUIDISTANT PERMUTATION ARRAYS

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Dedicated to Professor A. Kotzig on the occasion of his sixtieth birthday

An equidistant permutation array $A(r, \lambda; v)$ is a $v \times r$ array defined on a symbol set V such that each row of the array is a permutation of the symbol set V and any two distinct rows of A have precisely λ common column entries. Given r and λ, one can ask for the largest v such that an $A(r, \lambda; v)$ exists or find the smallest v such that an $A(r, \lambda; v)$ exists which cannot be extended to an $A(r, \lambda; v+1)$. In this paper, we consider arrays of the latter type. Such arrays we call maximal. Several classes of maximal equidistant permutation arrays are given and bounds are obtained.

1. Introduction

An equidistant permutation array (EPA) is a $v \times r$ array defined on a symbol set V such that

(1) every row is a permutation of the symbol set V,

(2) every pair of distinct rows has precisely λ common column entries.

We denote such an array by $A(r, \lambda; v)$.

An EPA, $A = A(r, \lambda; v)$ is called maximal if it is impossible to add a $(v+1)$st row to A such that the resulting array is an $A(r, \lambda; v+1)$. A maximal $A(r, \lambda; v)$ is denoted by $\bar{A}(r, \lambda; v)$. Define

$$R(r, \lambda) = \max\{v\colon \text{there exists an } A(r, \lambda; v)\}$$

and

$$\bar{R}(r, \lambda) = \min\{v\colon \text{there exists an } \bar{A}(r, \lambda; v)\}.$$

There are a number of results on the function $R(r, \lambda)$. For example, if $n = r - \lambda$ then

$$R(r, \lambda) \leqslant \max\{R(n+1, 1), 2 + \lfloor \lambda / \lceil \tfrac{1}{3}n \rceil \rfloor, n^2 - 5n + 7, \tfrac{1}{2}(n+2)^2\}.$$

This bound is dependent on the results for $R(r, 1)$. In this case, the following is known:

$$2r - 4 \leqslant R(r, 1) \leqslant r^2 - 4r + 1, \quad r \geqslant 6.$$

In many cases, the lower bound has been improved. For example, $R(r^2 + r + 1, 1) \geqslant r^3 + r^2$ whenever r is a prime or prime power. These and other results on $R(r, \lambda)$ can be found in [1, 2, 3, 8, 10, 11]. In this paper, we are interested in maximal EPAs. We require several more definitions.

*Research supported by NSERC under grant No. A9258.

A *generalized Room square* (GRS) is an $r \times r$ array defined on a symbol set V of cardinality v such that

(1) every cell contains a (possibly empty) subset of V,

(2) every element of V is contained in every row and column of the array precisely once, and

(3) every pair of distinct elements of V is contained in λ of the cells.

Such an array will be denoted by $S(r, \lambda; v)$.

An (r, λ)-design D is a collection B of subsets (blocks) taken from a finite set V of elements (varieties) such that

(1′) every element of V is contained in precisely r blocks of D,

(2′) every pair of distinct elements of V is contained in exactly λ blocks of D.

If each block of D has cardinality k, D has v varieties and b blocks, then D is called a balanced incomplete block design and is denoted (v, b, r, k, λ)-BIBD.

In Section 2, (r, λ)-designs are used to generalize a construction for maximal EPAs which was given in [6]. In Section 3, we show that this construction, applied to BIBDs, gives maximal EPAs.

We conclude this section with a remark which will be useful in Section 3. Let $A(r, \lambda; v)$ be an EPA with rows $r_1, r_2, \ldots, r_v$ where r_i is a permutation of the symbol set $V = \{1, 2, \ldots, r\}$. Form an $r \times r$ array S whose rows and columns are indexed by V. For each h, $1 \leqslant h \leqslant v$, in the cell $(r_h(j), j)$ place the symbol X_h, $1 \leqslant j \leqslant r$. It is readily seen that S is a GRS defined on the symbol set $\{X_i\colon 1 \leqslant i \leqslant v\}$. We call S the GRS *associated* with the EPA $A(r, \lambda; v)$. We will, at times, find it convenient to work with the GRS rather than the EPA.

2. EPAs and $(r, 1)$-designs

Let a be a permutation acting on the set $N = \{1, 2, 3, \ldots, n\}$. Define

$$E(a) = \{i \in N\colon a(i) \neq i\}.$$

For any $A \subseteq S_n$, $E(A) = \{E(a)\colon a \in A\}$. Let $L = \{l_1, l_2, \ldots, l_t\}$ be a set of positive integers. $A(n, L)$ is a set of permutations such that for all $a, b \in A$ $(a \neq b)$, $n - |E(a^{-1}b)| \in L$. In the case where $|L| = 1$, $A(n, L)$ is just an EPA.

Lemma 2.1. *Let $A = A(n, L)$ such that*

(i) $|E(a) \cap E(b)| \leqslant 1$ *for all $a, b \in A$, $a \neq b$.*

Then: (a) $|E(A)| = |A|$;

(b) $E(a^{-1}b) = E(a) \cup E(b)$ *for all a, $b \in A$, $a \neq b$, and hence $n - |E_1 \cup E_2| \in L$ for all $E_1, E_2 \in E(A)$, $E_1 \neq E_2$;*

(c) *A is a proper subset of some $A(n, L)$ with property* (i) *iff there exists $E' \subset \{1, 2, \ldots, n\}$ such that $|E'| \neq 1$, $E' \notin E(A)$ and $n - |E' \cup E| \in L$ for any $E \in E(A)$.*

Proof. (a) We have, of course $|E(A)| \leqslant |A|$ from definition of $E(A)$. Suppose $|E(A)| < |A|$. Hence $E(a) = E(b)$ for some a, $b \in A$, $a \neq b$. We obtain $E(a) \cap E(b) = E(a) = E(b)$; condition (i) implies $E(a) \leqslant 1$, $E(b) \leqslant 1$. But neither $E(a) = 1$ nor $E(b) = 1$ is possible. We have $|E(a)|=|E(b)|=0$, i.e. both a, b are the identity permutation. This contradicts the supposition $a \neq b$.

(b) For any $a, b \in S_n$ we have $E(a)\Delta E(b) \subseteq E(a^{-1}b) \subseteq E(a) \cup E(b)$ where $E(a)\Delta E(b) = (E(a)\cup E(b)) - (E(a)\cap E(b))$ is the symmetric difference of the sets $E(a)$, $E(b)$. In case $|E(a)\cap E(b)|=0$ we will have $E(a^{-1}b)=E(a)\cup E(b)$ immediately. Suppose now that $|E(a)\cap E(b)|=1$ and $E(a)\cap E(b)=\{p\}$ for some $p \in \{1,\dots,n\}$. We have $a(p)\neq p \neq b(p)$. The case $a(p)=b(p)$ will imply $a(a(p))\neq a(p)$, $b(a(p))\neq a(p)$, i.e. $a(p) \in E(a)\cap E(b)$ and $|E(a)\cap E(b)|>1$ which contradicts condition (i). Hence, $a(p)\neq b(p)$ and $E(a^{-1}b)= E(a)\cup E(b)$.

(c) This follows from (a) and (b).

In the next theorem we give a construction for a class of $A(n, L)$.

Theorem 2.1. *If there exists an $(r, 1)$-design D with n blocks and v varieties, then there exists an $A(n, n-2r+1; v)$.*

Proof. Let $T=\{B_1, B_2, \dots, B_n\}$ be the blocks of D. For each variety $x \in D$, let f_x be any permutation acting on T such that $E(f_x)=\{B\colon x \in B\}$. We now show that $A=\{f_x\colon x \in D\}$ is an $A(n, l; v)$. It is clear that A satisfies property (i) of Lemma 2.1. Hence, from (b) of Lemma 2.1, $E(a^{-1}b)=E(a)\cup E(b)$ for all $a, b \in A (a \neq b)$. But $E(a)=r$, for all $a \in A$ and, thus, $E(a^{-1}b)=2r-1$. This completes the proof.

This result generalizes a construction given in [6].

Corollary 2.1. *Let $A=A(n, l)$ be the permutation array corresponding to a given $(r, 1)$-design D. If G is the GRS corresponding to A, then the blocks of G having cardinality greater than one are precisely the blocks of the complement to D.*

Proof. A block B of G of cardinality $k \geqslant 2$ corresponds to a set $\{a_1, a_2, \dots, a_k\}$ permutations of A with the property that $a_i(b)=a$ for all i, $1 \leqslant i \leqslant k$ and fixed elements a and b. Since $k \geqslant 2$, and $t(a_i, a_j)=0$ for all i, j $(i \neq j)$, then $a \notin E(a_i)$, $1 \leqslant i \leqslant k$. Hence, the elements of D associated with $a_1, a_2, \dots, a_k$ do not occur in the block B' of D associated with b. Hence, the elements of D associated with $a_1, a_2, \dots, a_k$ form the block $\bar{B}'$. This completes the proof.

We remark that the same procedure applied to an (r, λ)-design will produce an $A(n, L)$ where $L=\{n-2r+\lambda+t\colon 0 \leqslant t \leqslant \lambda-1\}$. This result follows from the fact that for any two permutations a, $b (a \neq b)$ $|E(a^{-1}b)|=|E(a)|+|E(b)|-|E(a)\cap E(b)|-t(a, b)$ where $t(a, b)=|\{i \in E(a)\cap E(b)\colon a(i)=b(i)\}|$ and the fact that $0 \leqslant t(a, b) \leqslant \lambda-1$.

Theorem 2.1 shows that any $(r, 1)$-design with n blocks can be used to produce an $A(n, n-2r+1)$. Given an $A(n, l)$, when can we produce an $(r, 1)$-design. An $A(n, l)$ is said to have property (i′) if $|E(a)|=r=\frac{1}{2}(n+1-l)$, for all $a \in A$ and $|E(a)\cap E(b)|=1$ for all $a, b \in A$ $(a \neq b)$. It is easily seen that there exists an $(r, 1)$-design with n blocks if and only if there exists an $A(n, n-2r+1)$ having property (i′).

Lemma 2.2. *Let $A=A(n, l)$ have property (i′) and $A \cup \{a'\}$ be an $A(n, l)$ for some $a' \in S_n \setminus A$. Then*

(a) $|E(a')| \geqslant r-1$,
(b)

$$\begin{aligned} &|E(a')| = r-1 && \textit{iff } E(a') \cap E(a) = \emptyset, \textit{ for all } a \in A. \\ &|E(a')| = r && \textit{iff } |E(a') \cap E(a)| = 1, \textit{ for all } a \in A. \\ &|E(a')| = r+1 && \textit{iff } |E(a') \cap E(a)| = 2, \textit{ and } t(a, a') = 0 \textit{ for all } a \in A. \end{aligned}$$

(c) $|E(a')| \geqslant r+2$ *implies* $|E(a') \cap E(a)| \geqslant 2$, *and* $|E(a') \cap E(a)| + t(a, a') = |E(a') - r + 1 \geqslant 3$ *for all* $a \in A$.

Proof. Recall $|E(a^{-1}a')| = |E(a)| + |E(a')| - |E(a) \cap E(a')| - t(a, a')$. Since $|E(a)| = r$ for all $a \in A$ and $|E(a^{-1}b)| = n-l$ for all $a, b \in A$ $(a \neq b)$ then $|E(a')| = r-1 + |E(a) \cap E(a')| + t(a, a')$. (a) now follows.

(b) follows from the above and Lemma 1 using the fact $|E(a) \cap E(a')| \leqslant 1$ implies $t(a, a') = 0$. This completes the proof.

Recall that $A = A(n, l)$ is maximal and denote it by $\bar{A}(n, l)$ if $A \cup \{a'\} \neq A(n, l)$ for any $a' \in S_n \setminus A$. An $(r, 1)$-design D with v varieties and b blocks is extendible to an $(r, 1)$-design D' with $v+1$ varieties and b blocks if D is isomorphic to a restriction of D'. This is a special case of the extension given implicitly in (c) of Lemma 2.1. An $A = A(n, l)$ with property (i′) is strongly extendible if there exists an $a \in S_n \setminus A$ such that $A \cup \{a\}$ is an $A(n, l)$ with property (i′).

Theorem 2.2. *An $A(n, l)$ with property* (i′) *is strongly extendible iff the corresponding $(r, 1)$-design is extendible.*

Corollary 2.2. *An $A(n, l; v)$ with property* (i′) *is extendible to an $A(n, l; n)$ with property* (i′) *if $v > ((n-l-1)/2)^2$.*

Proof. This follows from a result on extendability of $(r, 1)$-designs and can be found in [8].

We conclude this section with the following useful lemmas on EPAs $A = A(n, l; v)$ with $|A| = v \geqslant 2$. Suppose that $|E(a)| = r$ for any $a \in A$. It is evident that $r \geqslant 2$.

Lemma 2.3. *Suppose that $|E(a)| = 2$ for any $a \in A$. Then either*
(a) $l = n-4$, $E(a) \cap E(b) = \emptyset$ *for any* $a, b \in A$, $a \neq b$; *or*
(b) $l = n-3$, $|\bigcap_{a \in A} E(a)| = 1$; or
(c) $l = n-3$, $A = \{(i, j), (i, k), (j, k)\}$ *for some* $1 \leqslant i < j < k \leqslant n$.

Proof. A consists of transpositions, i.e. cycles of length 2. The case (a) corresponds to the possibility that every pair of these distinct cycles has no element of $\{1, 2, \ldots, n\}$ in common. The only possibility is that every pair of these cycles has exactly one element in common. For this case Lemma (3.4) of [4] gives only two possibilities corresponding to cases (b) and (c).

Remark. Using known equalities [3], $\max|A(n, n-3)| = n-1$, $\max|A(n, n-4)| = \lfloor \frac{1}{2} n \rfloor$ we can show easily that

in the case (a) of Lemma 2.1, $A=\bar{A}(n, n-4)$ iff $v=\lfloor\frac{1}{2}n\rfloor$;
in the case (b) of Lemma 2.1, $A=\bar{A}(n, n-3)$ iff $v=n-1$.
Also, $A=\bar{A}(n, n-3)$ in the case (c) of Lemma 2.1.

Lemma 2.4. *Let $A=A(n, l; v)$. Then $l\neq n-1$ and $l\leqslant n-3$ for $|A|>2$.*

Proof. It follows trivially from Lemma 2.1. In fact, suppose $l=n-2$ and a_1, a_2 and a_3 are three distinct elements of A. Then $\{a_1^{-1}a_2, a_1^{-1}a_3\}$ is an $A(n, n-2; 2)$ with $|E(a_1^{-1}a_2)|=|A(a_1^{-1}a_3)|=2$, which is an impossibility.

Lemma 2.5. *Let $A=A(n, l; v)$, $v>2$ and let $a'\in A$. Denote $A'=\{b^{-1}a': b\in A, b\neq a'\}$; then*

(a) *$A'=A(n, l; v-1)$, with*
(b) *$|E(a)|=n-l$ for any $a\in A'$ and*
(c) *$|E(a)\cap E(b)|\geqslant\max(\frac{1}{2}(n-l), 2)$ for any $a, b\in A'$, $a\neq b$.*

Proof. (a) and (b) follow from the fact that Hamming distance $|E(a^{-1}b)|$on S_n is invariant of translation, i.e. $|E(a^{-1}b)|=|E((ac)^{-1}(bc))|=|E((ca)^{-1}(cb))|$.

Let $a, b\in A'$, $a\neq b$.

$$\begin{aligned} n-l=|E(a^{-1}b)| &\geqslant |E(a)\Delta E(b)| \\ &=|(E(a)\cup E(b))-E(a)\cap E(b)| \\ &=|E(a)|+|E(b)|-2|E(a)\cap E(b)| \\ &=2(n-l)-2|E(a)\cap E(b)|. \end{aligned}$$

Hence, $|E(a)\cap E(b)|\geqslant\frac{1}{2}(n-l)$. Moreover, $\frac{1}{2}(n-l)>1$ (because $l\leqslant n-3$) from Lemma 2.1 and $|E(a)\cap E(a)|$ is an integer; so (c) is proved.

3. Maximal EPAs

In this section, we find several classes of maximal EPAs. Let $\mathrm{LS}(n+l, l; n)$ be an $A(n+l, l; n)$ obtained from a latin square by adjoining l fixed points.

Theorem 3.1. *For each positive integer n and l, $\mathrm{LS}(n+l, l; n)$ is an $\bar{A}(n+l, l; n)$.*

Proof. Let G be the GRS associated with LS. The nonempty cells of G contain blocks of size 1 and n. Without loss of generality, assume that the blocks of size n occur in a $l\times l$ subarray S and S occurs in the upper left corner of G and L is the latin square subarray in the lower right corner. Suppose G is extendible by adjoining a new element x. If x is contained in all blocks of size n, x cannot occur in any cell of L. Hence, x cannot occur in any row or column of G which contains L. Suppose there is some block B of size n which does not contain x. x must occur once in the row of G which contains B. This row does not contain a row of L. Since x must occur with every element l times, x must occur in each row and column of L. This is impossible. Hence, G is not extendible and LS is maximal.

This theorem implies that $\bar{R}(n, l) \leqslant n-l$. We conjecture that $\bar{R}(n, l)=n-l$. In the next theorem we establish a lower bound on the size of $\bar{R}(r, 1)$.

Theorem 3.2. $\bar{R}(r, 1) \geqslant \frac{1}{3}(r+3)$.

Proof. Let A be an $A(r, 1; v)$ with $v<\frac{1}{3}(r+3)$. Consider the GRS associated with A. Denote this GRS by S, its symbol set by V and let l be the cardinality (size) of a block of maximum length in S.

Any element of S is contained in at most $v-1$ blocks of size greater than one and, hence, must occur in at least $r-v+1$ blocks of size one (singletons). Suppose B is a block of size l in $SV \setminus B=\{v_1, v_2, \ldots, v_{v-l}\}$ and B is contained in cell C_0 of S. Let C_i be a cell of S which contains v_i as a singleton and C_i does not occur in a row or column containing any other C_j $(j \neq i)$. That such a set of cells exists follows from the fact that as we select each C_i we eliminate at most 2 singletons for each element of $V \setminus B$ not yet chosen and, also, $r-v-1 \geqslant 2(v-l)$.

Permute rows and columns of S so that the cells C_i, $0 \leqslant i \leqslant v-l$, are contained in the $(v-l+1) \times (v-l+1)$ subarray in the upper left corner of the array. Consider the $(r-v+l-1) \times (r-v+l-1)$ subarray E which is contained in the lower right corner of S. We now select $r-v+l-1$ empty cells of E, $e_1, e_2, \ldots, e_{r-v+l-1}$ which cover the rows and columns of E. This can, of course, be thought of as selecting a matching in the bipartite graph formed from the rows and columns of E and edges corresponding to the empty cells. Each row and column in this subarray contains at least $r-v+l-1-v$ empty cells. P. Hall's theorem guarantees a matching (transversal) of empty cells provided for each subset X of the rows $|X| \leqslant |N(X)|$ where $N(X)$ is the set of columns having at least one empty cell in some row of X. We need only check sets X such that $|X| \geqslant r-2v+l-1$. Since $r-2v+l-1>V$, we know that if $|N(X)|<|X|$ then there must be a column of E having more than v nonempty cells which is impossible. Therefore, by Hall's theorem there is a transversal of empty cells in E. If we now add a new symbol to each of the cells C_i, $0 \leqslant i \leqslant v-l$, and the cells $e_1, e_2, \ldots,$ $e_{v-r+l-1}$, we get a GRS of side r and order $v+1$ and, hence, the EPA corresponding to the original GRS was not maximal. This completes the proof.

The above result leads one to ask whether every $(r, 1)$-design having $v \leqslant r-1$ varieties is the underlying design for some GRS. This is a similar question to that posed by Erdös, Faber, and Lovász [4] concerning the resolvability of $(r, 1)$-designs with $v \leqslant r$ elements. Erdös, Faber, and Lovász conjecture that any $(r, 1)$-design defined on $v \leqslant r$ varieties is resolvable. This conjecture is, to the authors' knowledge, unsolved. Partial results exist and the interested reader is referred to [9].

Another class of maximal EPAs is given in the next theorem.

Theorem 3.3. *If there exists a* $(v, b, r, k, 1)$*-BIBD* $(k \geqslant 3)$, *then there exists an* $\bar{A}(b, b-2r+1; v)$.

A proof of this theorem can be found in [9]. In this special case, where $v=b$, the result stated in Theorem 3.3 was obtained in [6].

4. $\bar{A}(n, l; n)$

We now consider a special class of maximal EPAs. An EPA $A(n, l; v)$ is called square if $n=v$. We are interested in $\bar{A}=A(n, l; n)$. For $l=0$, a latin square provides an example of an $\bar{A}(n, 0; n)$ for any n. For $l=q^2-q$, $n=q^2+q+1$ and, if there exists a finite projective plane, then there exists [6] an $\bar{A}(q^2+q+1, q^2-q; q^2+q+1)$.

It has been shown [1] that

$$|A(n, l)| \leqslant \max\{l+2, (n-l)^2+(n-l)+1\}.$$

If $(n-l)^2+(n-l)+1 \leqslant l$, then

$$|A(n, l)| \leqslant l+2$$

and, hence, from Lemma 2.4

$$|A(n, l)| < n$$

and the square cannot exist. This will occur provided $l \geqslant n-1-\sqrt{n+2}$. Let $f(n)$ be the maximum value of l such that an $A(n, l; n)$ exists.

Theorem 4.1. (a) $f(n) < n-1-\sqrt{n+2}$.

(b) *Let* $n=q^2+q+1$ *for* q *a prime power* p^a. *Then*

$$n-1-2\sqrt{n-1} < q^2-q \leqslant f(n) \leqslant q^2-1.$$

(c) $f(7)=2$.

Proof. Part (a) follows from the above. The lower bound in (b) follows from the results of [1]. The upper bound follows from (a), and the fact that

$$q^2+q-\sqrt{q^2+q+3} < q^2+q-\sqrt{q^2} = q^2.$$

In the case of $n=7$, $f(7)<3$ or $f(7) \leqslant 2$. From (b) of the theorem $f(7) \geqslant 2$. Hence, $f(7)=2$. This completes the proof.

We conjecture that the lower bound in (b) is the exact value of $f(n)$. Since there exists an $A(6, 1; 7)$ [3], there exists an $A(7, 2; 7)$ which is not obtained from a finite projective plane using Theorem 3.1. Thus, the extremal case for $f(2)$ is not unique. We remark, however, that the $A(7, 2; 7)$ just given is not maximal. Another example of a square having the parameters of a square constructed in [6] but not obtainable by this construction is an $A(43, 30; 43)$. To construct such an array by the result of [6] requires a finite projective plane of order 6. This array can be constructed as follows. By a construction given in [5], there exists an $A(2q-1, q-3; q(q-2))$. If we take $q=11$, we obtain an $A(21, 8; 99)$. Deleting 56 permutations and adding 22 fixed points, produces the required array.

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Received 8 November 1980.