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ON PERMUTATION ARRAYS, TRANSVERSAL SEMINETS AND RELATED STRUCTURES

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Exploiting some ideas of [4], this paper is focused on the equivalence between sets of mutually orthogonal permutation arrays and a special class of seminets (the so-called transversal seminets). Besides this equivalence, Section 2 contains a construction method for transversal seminets using groups. Nonsolvable permutation groups of prime degree and the projective special linear groups $PSL(2,2^m)$ yield examples for this method. In Section 3 some upper bounds are proved for the number of mutually orthogonal permutation arrays, depending on the intersection structure of these arrays. With the results of Sections 4 it is shown that all examples of Section 2 are row-extendible. Section 5 deals with several incidence structures associated to transversal seminets. The consequences are investigated when these incidence structures have special properties (for instance, when they are pairwise balanced designs). Section 6 discusses briefly the relations of transversal seminets with other mathematical structures, e.g. with transversal packings, generalized orthogonal arrays, and sets of mutually orthogonal partial quasigroups.

1. INTRODUCTION

A $v\times r$ matrix $A = (a_{ij})$ with entries a_{ij} from the set $\{1,2,\ldots,r\}$ is called a *permutation array* if each row of A contains each of the elements $1,2,\ldots,r$ exactly once, i.e. if the rows of A represent permutations of $\{1,2,\ldots,r\}$. The *intersection structure* of A is defined as the $v\times v$ matrix $F(A) = (F_{ii'}(A))$ with $F_{ii'}(A) := \{j \mid 1\le j\le r,\ a_{ij} = a_{i'j}\}$. The $v\times r$ permutation arrays $A = (a_{ij})$ and $B = (b_{ij})$ are called *similar* if $F(A) = F(B)$. Clearly, A and B are similar if and only if, for all indices i,i',j,

$$a_{ij} = a_{i'j} \iff b_{ij} = b_{i'j} .$$

The permutation arrays A and B are called *orthogonal* if they are similar and if, for all i,i',j,j',

$$a_{ij} = a_{i'j'} \text{ and } b_{ij} = b_{i'j'} \implies j = j' .$$

This concept of orthogonality generalizes the well-known idea of orthogonal latin rectangles. It has been defined and investigated by Bonisoli and Deza in [4].

(See also [7] for closely related considerations.)

Similarly as for latin rectangles and latin squares, one will be interested in sets $\mathcal{A} = \{A_1,A_2,\ldots,A_t\}$ of t mutually orthogonal permutation arrays, with t possibly greater than 2. In this case the *intersection structure* $F(\mathcal{A}) = (F_{ii'}(\mathcal{A}))$ of $\mathcal{A}$ is defined as the common intersection structure of the A_k, i.e. $F(\mathcal{A}) := F(A_1) = F(A_2) = \ldots = F(A_t)$.

A permutation array $A = (a_{ij})$ is called *standardized* if $a_{1j} = j$ for all j.

Without loss of generality, it will be assumed in this paper that each permutation array is standardized, and that it satisfies the following nontriviality conditions (C_1) *and* (C_2).

(C_1) The permutation array A has no constant column, i.e. each column of A contains at least two distinct values,

(C_2) any two rows of A are distinct.

Let X be a nonempty set, and let $L_0,L_1,\ldots,L_t$ (with $t \geq 1$) be mutually disjoint sets of nonempty subsets of X. The elements of X will be called *points* and the elements of $\bigcup_{k=0,1,\ldots,t} L_k$ *lines*. Then $\mathfrak{S} := (X;L_0,L_1,\ldots,L_t)$ is called a *seminet* or (more precisely) a $(t+1)$-*seminet* if

(S_1) any two distinct lines intersect in at most one point,

(S_2) each class L_i partitions the point set X.

Condition (S_2) justifies the term *parallel class* for each of the lines L_i. The notion of a seminet generalizes such well-known structures like affine planes, nets and (more generally) the parallel structures of André [2]. A subset of X is called a *transversal* of the seminet $\mathfrak{S}$ if it intersects each line of $\mathfrak{S}$ in exactly one point. If $\mathfrak{S}$ has a transversal consisting of r points, then each parallel class L_i contains exactly r lines (and hence each further transversal of the seminet consists also of exactly r points). If $T_1,T_2,\ldots,T_v$ are transversals of $\mathfrak{S}$ then $\mathfrak{T} := (X;L_0,L_1,\ldots,L_t;T_1,T_2,\ldots,T_v)$ is called a *transversal seminet* (or *transversal* $(t+1,r)$-*seminet* if each transversal consists of r points). Bonisoli and Deza [4] pointed out that there is a close relationship between sets of mutually orthogonal permutation arrays and other mathematical structures. For instance, they proved that each set of t mutually orthogonal $v\times r$ permutation arrays is equivalent to a 1-design with v treatments, replication number r and $t+1$ mutually orthogonal resolutions (see Section 5). Moreover, it was shown that any of these are equivalent to a transversal $(t+1,r)$-seminet with v transversals. Therefore many of the examples and results in this paper can be translated into analogous statements on combinatorial designs with mutually ortogonal resolutions.

2. AN EQUIVALENCE AND A CONSTRUCTION METHOD

Let $\mathcal{T} := (X;L_0,L_1,\dots,L_t;T_1,T_2,\dots,T_v)$ be a transversal seminet with $L_0 := \{l_0^1,l_0^2,\dots,l_0^r\}$. (In fact, throughout this paper the set of parallel classes, the set of transversals and the set of lines of L_0 of any transversal seminet are assumed to be linearly ordered, by the numbering of their elements.) One can now define t mutually orthogonal $v\times r$ permutation arrays $A_k = (a_{ij}^k)$, $k=1,2,\dots,t$, in the following way: For $i\in\{1,2,\dots,v\}$, $j\in\{1,2,\dots,r\}$, $k\in\{1,2,\dots,t\}$ let x be the unique point contained in $T_i \cap l_0^j$, and let l be the L_k-line through x. Let y be the unique point with $y\in T_1\cap l$, and let l_0^c be the L_0-line through y. Finally, define $a_{ij}^k := c$. From the properties of transversal seminets one can conclude that $\mathcal{A}(\mathcal{T}) := \{A_1,A_2,\dots,A_t\}$ is, in fact, a set of mutually orthogonal permutation arrays.

The conditions (C_1) and (C_2) of Section 1 can be paraphrased in terms of transversal seminets as follows:

(D_1) There is no point of the seminet which is contained in all transversals, i.e. $\bigcap_{i=1,2,\dots,v} T_i = \emptyset$. (This implies, in particular, $|l|\geq 2$ for each line l, and $v\geq 2$.)

(D_2) Any two transversals are distinct, i.e. $T_i \neq T_j$ for $i\neq j$.

In the construction procedure for $\mathcal{A}(\mathcal{T})$ only those points of X have been used which are contained in one of the transversals T_i. Therefore one can always restrict oneself to the *reduced* transversal seminet $(X';L_0',L_1',\dots,L_t';T_1,T_2,\dots,T_v)$ defined by $X' := \bigcup_{i=1,2,\dots,v} T_i$, $L_k' := \{l\cap X' \mid l\in L_k\}$.

In the rest of this paper all transversal seminets are assumed to be reduced and to satisfy the conditions (D_1) and (D_2).

The process of constructing mutually orthogonal permutation arrays from transversal seminets can be reversed: Let $\mathcal{A}$ be a set of t mutually orthogonal $v\times r$ permutation arrays $A_k = (a_{ij}^k)$, $k=1,2,\dots,t$. Define $Y := \{1,2,\dots,v\}\times\{1,2,\dots,r\}$, and let ψ be the equivalence relation on Y with $(i,j)\,\psi\,(i',j')$ if and only if $j=j'$ and $j\in F_{ii'}(\mathcal{A})$. Define the point set X of the seminet as the set of equivalence classes of ψ, i.e. $X := Y/\psi = \{[(i,j)]\psi \mid (i,j)\in Y\}$. For $c = 1,2,\dots,r$ and $k=1,2,\dots,t$ let $l_0^c := \{[(i,j)]\psi \mid j=c,\ i=1,2,\dots,v\}$ and $l_k^c := \{[(i,j)]\psi \mid a_{ij}^k = c\}$, and define $L_0 := \{l_0^1,l_0^2,\dots,l_0^r\}$, $L_k := \{l_k^1,l_k^2,\dots,l_k^r\}$. Finally, let $T_i := \{[(i,j)]\psi \mid j=1,2,\dots,r\}$, $i=1,2,\dots,v$. Then $\mathcal{T}(\mathcal{A}) := (X;L_0,L_1,\dots,L_t;T_1,T_2,\dots,T_v)$ is a (reduced) transversal seminet with $\mathcal{A}(\mathcal{T}(\mathcal{A})) = \mathcal{A}$. Summarizing, one obtains

2.1. PROPOSITION. *The existence of a set of* t *mutually orthogonal* $v\times r$ *permutation arrays is equivalent to the existence of a transversal* $(t+1,r)$*-seminet with* v *transversals.*

This equivalence was already observed in [4]. One can show even a little more. Let $\mathcal{T} = (X;L_0,L_1,\dots,L_t;T_1,T_2,\dots,T_v)$ and $\mathcal{U} = (Y;M_0,M_1,\dots,M_t;U_1,U_2,\dots,U_v)$ be reduced transversal seminets with $L_0 = \{l_0^1,l_0^2,\dots,l_0^r\}$ and $M_0 = \{m_0^1,m_0^2,\dots,m_0^r\}$, and assume $\mathcal{A}(\mathcal{T}) = \mathcal{A}(\mathcal{U})$. Define a mapping $\phi\colon X \to Y$ as follows. For $x \in T_i \cap l_0^j$ let $\phi(x)$ be the unique element contained in $U_i \cap m_0^j$. Then ϕ is an isomorphism of $\mathcal{T}$ and $\mathcal{U}$, i.e. ϕ is a bijection which maps parallel lines onto parallel lines (in fact, ϕ satisfies $\phi L_i = M_i$, $\phi T_i = U_i$ and $\phi l_0^i = m_0^i$ for all i). This yields

2.2. PROPOSITION. *If* $\mathcal{T}$ *and* $\mathcal{U}$ *are reduced transversal seminets with* $\mathcal{A}(\mathcal{T}) = \mathcal{A}(\mathcal{U})$ *then* $\mathcal{T}$ *and* $\mathcal{U}$ *are isomorphic.*

The transversal seminet $\mathcal{T} = (X;L_0,L_1,\dots,L_t;T_1,T_2,\dots,T_v)$ corresponds to a set $\mathcal{A}(\mathcal{T}) = \{A_1,A_2,\dots,A_t\}$ of mutually orthogonal *latin rectangles* exactly if $T_i \cap T_j = \emptyset$ for all i,j, $i\neq j$. The arrays $A_1,A_2,\dots,A_t$ form a set of mutually orthogonal *latin squares* if and only if $(X;L_0,L_1,\dots,L_t,L_{t+1})$, with $L_{t+1} := \{T_1,T_2,\dots,T_v\}$, is a net. In other words, the equivalence of Proposition 2.1 specializes to the classical correspondence of mutually orthogonal latin squares with nets.

The following theorem provides a construction method of sets of mutually orthogonal permutation arrays via seminets, using groups.

2.3. THEOREM. *Let* G *be a finite group with neutral element* e. *Let* t *and* s *be positive integers, and let* $S_0,S_1,\dots,S_t$ *and* $F_1,F_2,\dots,F_s$ *be nontrivial subgroups of* G *such that the following conditions are satisfied for all* $i,j \in \{0,1,\dots,t\}$, $k,l \in \{1,2,\dots,s\}$:

(1) $i \neq j \implies S_i \cap S_j = \{e\}$,

(2) $S_i \cap F_k = \{e\}$,

(3) $k \neq l \implies F_k \neq F_l$,

(4) $|F_k| = [G:S_i]$.

Then there exists a set of t *mutually orthogonal* $v\times r$ *permutation arrays, with* $r := |F_1|$ *and* $v := \frac{s}{r}\cdot|G|$.

<u>Proof</u>. For each $i \in \{0,1,\dots,t\}$ let L_i consist of the right cosets of S_i, i.e. $L_i = \{S_i g \mid g\in G\}$. Then $(G;L_0,L_1,\dots,L_t)$ is a $(t+1)$-seminet: Condition (S_1)

is trivially satisfied while (S_2) is a consequence of (1). Each right coset $F_k h$ of one of the subgroups F_k is a transversal of $(G;L_0,L_1,\dots,L_t)$: It has to be shown that $|S_i g \cap F_k h| = 1$ for all $g,h \in G$. As a consequence of (2) one obtains $|S_i g \cap F_k h| \le 1$. Assumption (4) then implies $\bigcup_{f \in F_k} S_i f = G$, and hence $|S_i g \cap F_k h| \ge 1$. Each transversal has $r = |F_1|$ elements, and each F_k has $\frac{|G|}{r}$ distinct right cosets. Thus there are $v = \frac{s}{r}\cdot|G|$ distinct transversals of the form $F_k h$, with $k \in \{1,2,\dots,s\}$ and $h \in G$. Finally, the nontriviality conditions (D_1) and (D_2) are a consequence of the nontriviality of the subgroups F_k and of (3), respectively. By Proposition 2.1, the proof is complete. □

The seminet $(G;L_0,L_1,\dots,L_t)$ of the above proof is, in fact, a *translation seminet*, i.e. it has a translation group operating regularly on its points: In the right regular representation of G each mapping $x \mapsto xg$, $g \in G$, maps every line onto a parallel line. On the other hand, each translation seminet can be obtained in this way from a group G and subgroups $S_0,S_1,\dots,S_t$ satisfying condition (1). Analogous group theoretic characterizations have been given, for instance, for translation planes, translation nets, translation structures and translation group divisible designs (see e.g. [1], [15], [22], [3] and [20]). Marchi [18] uses similar ideas for his characterization of regular affine parallel structures by partition loops. Probably one can formulate an analogue of Theorem 2.3 using loops instaed of groups. The problem would be to find examples for such a generalization. The rest of this section yields two classes of examples for Theorem 2.3. Cf. Huppert [14] and Wielandt [24] for the group theoretic notations.

2.4. EXAMPLE. Let G be a nonsolvable transitive permutation group of prime degree p. Let $v := p^2$, $r := \frac{|G|}{p}$, and let d be the positive integer with $d \le p-1$ and $d \equiv r \pmod p$. Then one can construct a set of $t := \frac{r}{d} - 1$ mutually orthogonal $v \times r$ permutation arrays: Assume G to operate on $\{a_1,a_2,\dots,a_p\}$. For each $k \in \{1,2,\dots,p\}$ define F_k to be the stabilizer of a_k in G, i.e. $F_k := G_{a_k}$. Then $F_k \ne F_1$ for $k \ne 1$ since G is doubly transitive (cf. Theorem 11.7 of [24]). Let $S_0,S_1,\dots,S_{t'}$ be the Sylow p-subgroups of G (with $t' \ge 1$ because G is nonsolvable). Obviously, these subgroups satisfy the assumptions of Theorem 2.3. Hence there are t' mutually orthogonal $v \times r$ permutation arrays. It remains to show $t' = t$ or, equivalently, that G has exactly $\frac{r}{d}$ Sylow p-subgroups. Let P be a Sylow p-subgroup of G. The only Sylow p-subgroup of the normaliser $N_G(P)$ of P is P itself. Hence $N_G(P)$ is solvable and thus of order $p \cdot d'$ with $d' \mid p-1$ (cf. [14], Satz II.3.6). Therefore the number n of Sylow p-subgroups satisfies $n = [G:N_G(P)] = \frac{G}{p \cdot d'} = \frac{r}{d'}$. From $n \equiv 1 \pmod p$ one obtains $d' \equiv r \pmod p$, i.e. $d = d'$ and $n = \frac{r}{d}$.

The nonsolvable transitive permutation groups of prime degree have been completely determined, due to the classification of finite simple groups (see

Corollary 4.2 of Feit [11]).

Notice that *solvable* transitive permutation groups of prime degree p cannot be used in the above construction: These groups have exactly one Sylow p-subgroup, which would imply $t=0$.

2.5. EXAMPLE. For each integer $m \geq 2$ one can construct a set of t mutually orthogonal $v \times r$ permutation arrays, with $t := 2^{m-1}(2^m-1)-1$, $v := (2^m+1)^2$ and $r := 2^m(2^m-1)$: Regard the projective special linear group $G = PSL(2,q)$, with $q = 2^m$, as a permutation group operating canonically on the $q+1$ points $\{a_1, a_2, \ldots, a_{q+1}\}$ of the projective line over the q-element field. For each $k \in \{1,2,\ldots,q+1\}$ define F_k to be the stabilizer of a_k in G, i.e. $F_k := G_{a_k}$. Let $S_0, S_1, \ldots, S_{t'}$ be the (mutually conjugate) cyclic subgroups of G of order $q+1$. By the results in [14], pp. 191-193, these subgroups satisfy the assumptions of Theorem 2.3. For the number t' of conjugates of S_0 one obtains $t'+1 = [G:N_G(S_0)] = \frac{(q-1)q(q+1)}{2(q+1)} = \frac{q(q-1)}{2} = t+1$, i.e. $t' = t$.

Notice that Hartman [12] used some of the groups $PSL(2,q)$ in order to construct designs with mutually orthogonal resolutions. For instance, for each $q \in \{19,31,43\}$ there exists a design with $v = q+1$ treatments, $r = \frac{q(q-1)}{6}$ replications and $t+1 = \frac{q-1}{6}$ mutually orthogonal resolutions.

3. BOUNDS FOR THE NUMBER OF MUTUALLY ORTHOGONAL PERMUTATION ARRAYS

A set $\{A_1, A_2, \ldots, A_t\}$ of mutually orthogonal permutation arrays is called *maximal* if there exists no permutation array A_{t+1} which is orthogonal to all A_k, $k = 1,2,\ldots,t$. A transversal seminet $(X; L_0, L_1, \ldots, L_t; T_1, T_2, \ldots, T_v)$ is called *L-maximal* if there exists no additional parallel class L_{t+1} such that $(X; L_0, L_1, \ldots, L_t, L_{t+1}; T_1, T_2, \ldots, T_v)$ is again a transversal seminet. The following lemma is obvious.

3.1. LEMMA. *A set $\mathcal{A}$ of mutually orthogonal permutation arrays is maximal if and only if the associated transversal seminet $\mathcal{T}(\mathcal{A})$ is L-maximal.*

3.2. PROPOSITION. *Each set $\mathcal{A}$ of mutually orthogonal permutation arrays of Example 2.5 is maximal.*

Proof. Let $G = PSL(2,2^m)$ be the group used for the construction of $\mathcal{A}$. The associated transversal seminet $\mathcal{T}$ has G as point set. The subgroups $S_0, S_1, \ldots, S_t$ are exactly the lines through the neutral element e of G, and the subgroups $F_1, F_2, \ldots, F_{q+1}$ are exactly the transversals through e (cf. the proof of Theorem 2.3). By Satz II.8.5 of Huppert [14], these subgroups cover G. Hence there cannot be any additional line through e, and $\mathcal{T}$ is therefore L-maximal. By Lemma 3.1 the proof is complete, since Proposition 2.2 yields $\mathcal{T} \cong \mathcal{T}(\mathcal{A})$. □

Actually, the assertion of Proposition 3.2 depends only on the intersection structure $F(\mathcal{A})$ of $\mathcal{A}$: Let $\mathcal{B} = \{B_1, B_2, \ldots, B_{t'}\}$ be a set of mutually orthogonal permutation arrays with $F(\mathcal{B}) = F(\mathcal{A})$. The construction procedure of Section 2 shows that the transversal seminets $\mathcal{T}(\mathcal{B})$ and $\mathcal{T}(\mathcal{B})$ have the same point sets and the same transversals. The transversal seminet $\mathcal{T}$ of the proof of Proposition 3.2 contains $n := |S_0| = [G:F_1]$ pairwise disjoint transversals $F_1 s$, $s \in S_0$, with $\bigcup_{s \in S_0} F_1 s = G$. Therefore each line of $\mathcal{T}$ contains exactly n points, and $\mathcal{T} \cong \mathcal{T}(\mathcal{A})$ implies that the same is true for $\mathcal{T}(\mathcal{A})$ and also for $\mathcal{T}(\mathcal{B})$. As a consequence, there is a point x of $\mathcal{T}(\mathcal{B})$ such that the number $t'+1$ of lines of $\mathcal{T}(\mathcal{B})$ through x cannot exceed the number $t+1$ of lines of $\mathcal{T}$ through e (in fact, this is true for *each* point x of $\mathcal{T}(\mathcal{B})$). Therefore Proposition 3.2 can be improved as follows.

3.3. PROPOSITION. *Let* $\mathcal{A} = \{A_1, A_2, \ldots, A_t\}$ *be one of the sets of mutually orthogonal permutation arrays of Example 2.5. Let* $\mathcal{B} = \{B_1, B_2, \ldots, B_{t'}\}$ *be a set of mutually orthogonal permutation arrays with* $F(\mathcal{B}) = F(\mathcal{A})$. *Then* $t' \leq t$.

The next lemma gives an upper bound for the number of mutually orthogonal permutation arrays depending on the intersection structure of the arrays. The proof of this lemma is a unified version of the proofs of several related results in [4] and [7].

3.4. LEMMA. *Let* $\mathcal{A} = \{A_1, A_2, \ldots, A_t\}$ *be a set of mutually orthogonal* $v \times r$ *permutation arrays, and let* $I \subseteq \{1,2,\ldots,v\}$, $i_0 \in \{1,2,\ldots,v\}$, $J \subseteq \{1,2,\ldots,r\}$, $j_0 \in \{1,2,\ldots,r\}$ *satisfy*

a) $I \neq \emptyset$,

b) $\forall i_1, i_2 \in I: \ j_0 \in F_{i_1 i_2}(\mathcal{A})$,

c) $\forall i \in I: \ j_0 \notin F_{i i_c}(\mathcal{A})$,

d) $\forall j \in J \ \ \exists i \in I: \ j \in F_{i i_0}(\mathcal{A})$.

Then $t \leq r - |J| - 1$.

<u>Proof</u>. Let $A = (a_{ij})$ be a $v \times r$ permutation array with $F(A) = F(\mathcal{A})$. By a) there exists an element $i_1 \in I$. Define $c := a_{i_1 j_0}$. From b) one obtains $a_{ij_0} = c$ for all $i \in I$, and c) implies $a_{i_0 j_0} \neq c$. For each $j \in J$ there exists an $i \in I$ with $a_{i_0 j} = a_{ij}$, by d). Thus $j \neq j_0$ and $a_{ij} \neq a_{ij_0} = c$. Hence $a_{i_0 j} \neq c$. There-

fore $a_{i_0j} = c$ for exactly one element j of the $(r-|J|-1)$-element set $\{1,2,\ldots,r\} \setminus (J \cup \{j_0\})$. In particular, this is true for each of the permutation arrays $A_k \in \mathcal{A}$ with c and j replaced by c_k and j_k. By orthogonality, the mapping $k \mapsto j_k$ is injective. This implies $t \leq r - |J| - 1$. □

The following corollary translates Lemma 3.4 into the language of transversal seminets. Notice that this corollary could have been used in order to prove the Propositions 3.2 and 3.3.

3.5. COROLLARY. *Let* $\mathcal{T} = (X;L_0,L_1,\ldots,L_t;T_1,T_2,\ldots,T_v)$ *be a transversal seminet, and let* $I \subseteq \{1,2,\ldots,v\}$, $i_0 \in \{1,2,\ldots,v\}$ *and* $x \in X$ *satisfy*

a) $I \neq \emptyset$,

b) $x \in \bigcap_{i \in I} T_i$,

c) $x \notin T_{i_0}$.

Then $t \leq r - \delta - 1$, *with* $r := |T_1|$ *and* $\delta := |T_{i_0} \cap (\bigcup_{i \in I} T_i)|$.

3.6. COROLLARY. *Let* $\mathcal{A} = \{A_1,A_2,\ldots,A_t\}$ *be a set of mutually orthogonal* $v \times r$ *permutation arrays, and let* $\mu := \max\{|F_{ii'}(\mathcal{A})| \mid i,i'=1,2,\ldots,v,\ i \neq i'\}$. *Then* $t \leq r - \mu - 1$.

Proof. Choose $i, i_0 \in \{1,2,\ldots,v\}$ and $j_0 \in \{1,2,\ldots,r\}$, with $|F_{ii_0}(\mathcal{A})| = \mu$ and $j_0 \notin F_{ii_0}(\mathcal{A})$. Define $I := \{i\}$ and $J := F_{ii_0}(\mathcal{A})$. Then I, i_0, J, j_0 satisfy the assumptions of Lemma 3.4. □

3.7. COROLLARY. *Let* $\mathcal{A} = \{A_1,A_2,\ldots,A_t\}$ *be a set of mutually orthogonal* $v \times r$ *permutation arrays. Let* $\lambda := \min\{|F_{ii'}(\mathcal{A})| \mid i,i'=1,2,\ldots,v\}$, *and assume* $\lambda \geq 1$. *Then* $t \leq r - \lambda - 2$.

Proof. Choose $i_0, i_1, i_2 \in \{1,2,\ldots,v\}$, $i_1 \neq i_2$, and $j_0 \in \{1,2,\ldots,r\}$ with $j_0 \in F_{i_1i_2}(\mathcal{A})$, $j_0 \notin F_{i_1i_0}(\mathcal{A})$ and $j_0 \notin F_{i_2i_0}(\mathcal{A})$. Define $I := \{i_1,i_2\}$, $J := F_{i_1i_0}(\mathcal{A}) \cup F_{i_2i_0}(\mathcal{A})$. Then I, i_0, J, j_0 satisfy the assumptions of Lemma 3.4. Hence the proof is complete in the case $|J| \geq \lambda+1$. Assume now $|J| = \lambda$. Then $\lambda = |F_{i_1i_0}(\mathcal{A})| = |F_{i_2i_0}(\mathcal{A})|$, and thus $|F_{i_1i_2}(\mathcal{A})| = |J \cup \{j_0\}| = \lambda+1$. Therefore the case $|J| = \lambda$ is settled by Corollary 3.6. □

The Corollaries 3.5, 3.6 and 3.7 yield slight generalizations for some of the results in [4], [7] and [17] (which are formulated in terms of designs with mutually orthogonal resolutions).

A $v \times r$ permutation array A is called *row-transitive* if the rows of A form a set of permutations operating transitively on the set $\{1,2,\ldots,r\}$.

3.8. **PROPOSITION.** *Let* $\mathcal{A} = \{A_1, A_2, \ldots, A_t\}$ *be a set of mutually orthogonal* $v \times r$ *permutation arrays. Assume one of these arrays (and hence all of them) to be row-transitive. Then* $t \leq \min\{r-1, m+1\}$, *with* m *the largest number of mutually orthogonal latin squares of order* r.

Proof. Row-transitivity implies each line of the associated transversal seminet $\mathcal{T}(\mathcal{A}) = (X; L_0, L_1, \ldots, L_t; T_1, T_2, \ldots, T_v)$ to have exactly r points, since any line $l \in L_i$, $i \geq 1$, intersects each line of L_0 exactly once. Thus $(X; L_0, L_1, \ldots, L_t)$ is a $(t+1)$-net of order r. The existence of such a net is equivalent to the existence of $t-1$ mutually orthogonal latin squares of order r. Hence $t-1 \leq m$. A complete $(t+1)$-net of order r has exactly $r+1$ parallel classes. But a net with a transversal cannot be complete. Therefore $t+1 < r+1$. □

4. EXTENSION BY ROWS

A set $\mathcal{A} = \{A_1, A_2, \ldots, A_t\}$ of mutually orthogonal $v \times r$ permutation arrays is called *row-extendible* if it is possible to adjoin a new row to each of the arrays such that the resulting $(v+1) \times r$ permutation arrays are again mutually orthogonal.

A transversal seminet $(X; L_0, L_1, \ldots, L_t; T_1, T_2, \ldots, T_v)$ is called *transversal-extendible* if there exists a transversal seminet $(Y; M_0, M_1, \ldots, M_t; T_1, T_2, \ldots, T_v, T_{v+1})$ with $Y \supseteq X$ and $L_i = \{m \cap X \mid m \in M_i\}$. As transversal-extendibility is the obvious translation of row-extendability into the language of transversal seminets, one obtains

4.1. **LEMMA.** *A set* $\mathcal{A}$ *of mutually orthogonal permutation arrays is row-extendible if and only if the associated transversal seminet* $\mathcal{T}(\mathcal{A})$ *is transversal-extendible.*

The above definition of row-extendability is strictly stronger than the one given in [4] where the resulting arrays were only assumed to be similar. Both definitions coincide if $Y = X$ in the transversal seminets used in the definition of transversal-extendibility. In particular, the definitions coincide if $(X; L_0, L_1, \ldots, L_t)$ is a net (cf. Proposition 4.4 of [4]). In the case of mutually orthogonal latin squares, row-extendibility is equivalent to the existence of a common *column-transversal* (i.e. a usual transversal of latin squares with the condition "no two cells are on the same row" replaced by the weaker condition "not all cells are on the same row"); see [4], Proposition 4.2. Let $(X; L_0, L_1, \ldots, L_t; T_1, T_2, \ldots, T_v)$ be the transversal seminet associated to the latin squares $A_1, A_2, \ldots, A_t$.

Clearly, the latin squares have a common column-transversal exactly if there exists a transversal T_{v+1} of $(X;L_0,L_1,\dots,L_t)$ with $T_{v+1}\notin L_{t+1}:=\{T_1,T_2,\dots,T_v\}$. There is no such T_{v+1} if the net $(X;L_0,L_1,\dots,L_t,L_{t+1})$ is an affine plane (i.e. if $A_1,A_2,\dots,A_t$ form a complete set of mutually orthogonal latin squares). In general, it is an open question whether such a transversal exists if $(X;L_0,L_1,\dots,L_{t+1})$ is a *transversal-free* net in the sense of Dow [10].

Although the next proposition is easy to prove, the result is quite surprising.

4.2. PROPOSITION. *All sets $\mathcal{A}$ of mutually orthogonal permutation arrays of the Examples 2.4 and 2.5 are row-extendible.*

Proof. Let G be the group used in Example 2.4 or 2.5 for the construction of $\mathcal{A}$, and let $\mathcal{T}=(X;L_0,L_1,\dots,L_t;T_1,T_2,\dots,T_v)$ be the transversal seminet associated to G and the subgroups $S_0,S_1,\dots,S_t$ and $F_1,F_2,\dots,F_s$ of G. Recall from the proof of Theorem 2.3 that $X=G$, $L_i=\{S_ig\mid g\in G\}=\{S_if\mid f\in F_1\}$ for all $i\in\{0,1,\dots,t\}$, and $\{T_1,T_2,\dots,T_v\}=\{F_kg\mid g\in G,\ k=1,2,\dots,s\}$. By Lemma 4.1 and because of $\mathcal{T}\cong\mathcal{T}(\mathcal{A})$, it is sufficient to show that the transversal seminet $\mathcal{T}$ is transversal-extendible. Let $\overline{F}_1=\{\overline{f}\mid f\in F_1\}$ be a copy of F_1 with $\overline{F}_1\cap X=\emptyset$, and define $Y:=X\cup\overline{F}_1$. As all subgroups S_i, $i=0,1,\dots,t$, are conjugate to each other, there exist $a_0,a_1,\dots,a_t\in G$ with $S_i=a_iS_0a_i^{-1}$. Let $M_i:=\{S_ia_if\cup\{\overline{f}\}\mid f\in F_1\}$, $i=0,1,\dots,t$, and define $T_{v+1}:=\overline{F}_1$. Trivially then $Y\supseteq X$ and $L_i\supseteq\{m\cap X\mid m\in M_i\}$. In order to show $L_i=\{m\cap X\mid m\in M_i\}$, assume $f_1,f_2\in F_1$ and $S_ia_if_1=S_ia_if_2$. Then $(a_iS_0a_i^{-1})a_if_1=(a_iS_0a_i^{-1})a_if_2$ and thus $f_1f_2^{-1}\in S_0$. From $S_0\cap F_1=\{e\}$ one obtains $f_1=f_2$. This implies $|L_i|=|\{m\cap X\mid m\in M_i\}|$, and therefore $L_i=\{m\cap X\mid m\in M_i\}$. The same argument implies condition (S_2) for $(Y;M_0,M_1,\dots,M_t)$. It remains to show (S_1). Let $i,j\in\{0,1,\dots,t\}$, $f_1,f_2\in F_1$ and $x,y\in X$ satisfy $i\neq j$, $x\neq y$ and $x,y\in(S_ia_if_1\cup\{\overline{f}_1\})\cap(S_ja_jf_2\cup\{\overline{f}_2\})$. As $(X;L_0,L_1,\dots,L_t)$ is a seminet, either x or y cannot be contained in X. Assume $y\notin X$. Then $y=\overline{f}_1=\overline{f}_2$ and $x\in S_ia_if_1\cap S_ja_jf_2=(a_iS_0a_i^{-1})a_if_1\cap(a_jS_0a_j^{-1})a_jf_1=a_iS_0f_1\cap a_jS_0f_1$. This implies $xf_1^{-1}\in a_iS_0\cap a_jS_0$. Hence $a_iS_0=a_jS_0$ and thus $S_i=S_j$, contradicting $i\neq j$. □

The alternating group A_5 yields an example for some of the results of the preceding sections. Since A_5 and $PSL(2,2^2)$ are isomorphic as permutation groups, both of the Examples 2.4 and 2.5 imply the existence of a set $\mathcal{A}$ of 5 mutually orthogonal 25×12 permutation arrays. By Proposition 3.2, the set $\mathcal{A}$ is maximal, and by Proposition 4.2 $\mathcal{A}$ is row-extendible, i.e. there exists a set $\mathcal{A}'$ of 5 mutually orthogonal 26×12 permutation arrays. It is unknown whether $\mathcal{A}'$ is row-extendible or not.

5. TRANSVERSAL SEMINETS CARRYING DESIGNS

The concept of transversal seminets comprises a large variety of different mathematical structures. For detailed investigations one has therefore to add further restrictions. Sections 2 and 4 and parts of Section 3 treated transversal seminets with a group of translations operating regularly on the points. In this section additional assumptions will be imposed on the following incidence structures which are associated to each transversal seminet $\mathcal{T} = (X;L_0,L_1,\ldots,L_t;T_1,T_2,\ldots,T_v)$ (let $L := L_0 \cup L_1 \cup \ldots \cup L_t$ and $T := \{T_1,T_2,\ldots,T_v\}$, i.e. L is the set of lines and T the set of transversals of $\mathcal{T}$):

$I(X,L)$ — the treatments of this incidence structure are the points and the blocks are the lines of $\mathcal{T}$,

$I(T,X)$ — the treatments are the transversals and the blocks are the points of $\mathcal{T}$,

$I(X,L \cup T)$ — the treatments are the points and the blocks are the lines *and* the transversals of $\mathcal{T}$.

The incidence structure $I(X,L)$ is essentially the seminet of $\mathcal{T}$. In all three cases the incidence is defined naturally. For example, $x \in X$ and $T_i \in T$ are incident in $I(T,X)$ exactly if $x \in T_i$. $I(T,X)$ is the design with mutually orthogonal resolutions mentioned in Section 1: The blocks $x,y \in X$ are defined to be parallel in the i th parallel class if and only if there exists a line $l \in L_i$ with $x,y \in l$. Two interesting cases discussed later in this section occur when $I(T,X)$ or $I(X,L \cup T)$ are PBD's (pairwise balanced designs). For instance, $I(X,L \cup T)$ has this property if the associated set of permutation arrays is a complete set of mutually orthogonal latin rectangles (the examples after Proposition 5.3 show that the converse of this statement is not true). Before going into detail, some definitions are necessary.

The seminet $\mathcal{S} = (X;L_0,L_1,\ldots,L_t)$ is called n-*regular* if each line contains exactly n points. In this case $\mathcal{S}$ is a $(t+1,r)$-seminet with $r = |X|/n$ (i.e. $|L_i| = r$ for all i), and $\mathcal{S}$ is also called an (r,n)-*Mano configuration*. One has $n \leq r$, with equality if and only if $\mathcal{S}$ is a net or, equivalently, if the associated permutation arrays are row-transitive (see Section 3). An n-regular transversal seminet satisfies $n \leq v$ (because $rv = \Sigma_{i=1,2,\ldots,v}|T_i| \geq |X| = rn$), with equality if and only if the T_i's are pairwise disjoint. In this situation the set of transversals can be considered as a new parallel class L_{t+1}, and the associated permutation arrays are latin rectangles. Therefore, if $n = r = v$, then $I(X,L \cup T)$ is a $(t+2,r)$-net, and the associated permutation arrays are latin squares.

The transversal seminet $\mathcal{T} = (X;L_0,L_1,\ldots,L_t;T_1,T_2,\ldots,T_v)$ is called q- *uniform* if each point is contained in exactly q transversals. A q-uniform transversal seminet is n-regular with $n = v/q$. Each column of each of the associated permutation arrays contains each element $i \in \{1,2,\ldots,r\}$ either q or 0 times. An example of an n-regular (with n=4) but *not* q-uniform transversal seminet is provided by the orthogonal permutation arrays

$$A_1 = \begin{pmatrix} 1&2&3&4\\ 2&1&4&3\\ 3&4&1&2\\ 4&3&2&1\\ 1&3&4&2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1&2&3&4\\ 4&3&2&1\\ 2&1&4&3\\ 3&4&1&2\\ 1&4&2&3 \end{pmatrix}.$$

(The set $\{A_1,A_2\}$ is a row-extension of two orthogonal latin squares. It is easy to check that A_1,A_2 have no orthogonal mate.)

For each point x of a transversal seminet, let $\tilde{x}$ denote the number of transversals through x, and define $\tilde{X} := \{\tilde{x} \mid x \in X\}$. For each transversal seminet the incidence structure $I(T,X)$ is a 1-design $S_r(1,\tilde{X},v)$, i.e. each of the v treatments $T_i \in T$ is incident with exactly r blocks, and each block x is incident with $\tilde{x} \in \tilde{X}$ treatments. Notice that $I(T,X)$ may have repeated blocks. The blocks of $I(T,X)$ have $t+1$ mutually orthogonal resolutions (i.e. any two parallel classes of distinct resolutions coincide in at most one block). The resolutions are given by $L_0,L_1,\ldots,L_t$.

If there is a nonnegative integer λ with $|T_i \cap T_j| = \lambda$ for all i,j with $i \neq j$ (i.e. if the transversals form an (r,λ)-equidistant code), then $I(T,X)$ becomes a PBD with any two distinct treatments contained in exactly λ blocks. Equivalently, the associated permutation arrays are *equidistant* with Hamming-distance $r-\lambda$, i.e. any two rows of each of the permutation arrays coincide in exactly λ positions. If, moreover, the transversal seminet is q-uniform, then $I(T,X)$ is a 2-design $S_\lambda(2,q,v)$. As a consequence, $r(q-1) = \lambda(v-1)$. Analogously, if $I(T,X)$ is an $S_{\lambda'}(t',q,v)$, with $t' \geq 2$, then $r\binom{q-1}{t'-1} = \lambda'\binom{v-1}{t'-1}$. In this case, any t' distinct rows of each of the permutation arrays coincide in exactly λ' positions.

The transversal seminets constructed in the proof of Theorem 2.3 are n-regular and q-uniform, with $n = \frac{|G|}{r}$ and $q = s$. For none of these examples $I(T,X)$ is a PBD.

Many examples of transversal seminets are given in [17], with $I(T,X)$ an $S_{\lambda'}(t',q,v)$ or a group-divisible design (in this case $|T_i \cap T_j| \leq 1$ for all distinct i,j). For instance, by Theorem 1.5 of [17] there is a positive integer v_1 such that, for $v \geq v_1$, the condition $v \equiv 3 \pmod{12}$ is equivalent to the existence of a transversal seminet with $t = 2$ and the property that $I(T,X)$ is an $S(2,3,v)$. There are similar results of other authors; some of them are listed here:

1) Dinitz [8]. For each prime power of the form $q = 2^k c + 1$, with k a positive integer and $c > 1$ an odd integer, there exists a transversal seminet such that $t = c-1$ and $I(T,X)$ is an $S(2,2,q+1)$.

2) Kramer et al. [16]. There exists a transversal semient such that $t = 12$ and $I(T,X)$ is an $S(5,8,24)$.

3) Hartman [12]. There exist transversal seminets such that $I(T,X)$ is an $S(3,4,v)$ for $(v,t) = (20,3),(32,5),(44,7)$.

Transversal seminets can be represented by the following $(t+1)$-dimensional array of side r: the cell $(i_0,i_1,\ldots,i_t)$ contains the set $\{T_i \in T \mid x \in T_i\}$ if $l_0^{i_0} \cap l_1^{i_1} \cap \ldots \cap l_t^{i_t} = \{x\}$, and it is empty otherwise (the lines of each parallel class are assumed to be linearly ordered, i.e. $L_k = \{l_k^1, l_k^2, \ldots, l_k^r\}$). If $I(T,X)$ is an $S(2,2,v)$, then this array is called a Room $(t+1)$-cube of side $r = v-1$. Each such array is equivalent to $t+1$ pairwise orthogonal symmetric latin squares of size $(v-1)\times(v-1)$ (see [9]). If $I(T,X)$ is an $S_{\lambda'}(t',q,v)$, then the above $(t+1)$-dimensional array is a $(t+1)$-dimensional Room design; for instance, $S(5,8,24)$ yields a 13-dimensional Room design of side 253, see [16].

The incidence structure $I(X,L)$ of the seminet $\mathcal{S} = (X;L_0,L_1,\ldots,L_t)$ is a 1-design $S_{t+1}(1,\{|l| \mid l \in L\},|X|)$ without repeated blocks. If $\mathcal{S}$ is n-regular, then $I(X,L)$ becomes an $S_{t+1}(1,n,rn)$. $\mathcal{S}$ is called an *André seminet* (or *affine parallel structure*) if $I(X,L)$ is a linear space, i.e. if any two distinct treatments of $I(X,L)$ are incident with exactly one block. Obviously, André seminets do not admit transversals. A transversal seminet is called *almost-André* (or *complete*) if any two distinct points are either connected by a line or by a transversal. As each almost-André transversal seminet is L-maximal, Lemma 3.1 yields

5.1. REMARK. *Let the transversal seminet* $\mathcal{T}$ *be almost-André. Then the associated set* $\mathcal{A}(\mathcal{T})$ *of mutually orthogonal permutation arrays is maximal.*

Actually, Proposition 3.2 was proved essentially by showing that the involved transversal seminets are almost-André.

Recall that a set of t mutually orthogonal latin rectangles of size $v\times r$ is called *complete* if $t = r-1$.

5.2. PROPOSITION. *Let* $\mathcal{T}$ *be a transversal* $(t+1,r)$*-seminet with* v *transversals. Then* $t \le r-1$, *with equality if and only if the associated set* $\mathcal{A}(\mathcal{T})$ *of permutation arrays is a complete set of mutually orthogonal latin rectangles. In this case* $I(X,L\cup T)$ *is an* $S(2,\{r,v\},rv)$.

Proof. The inequality $t \le r-1$ is a trivial consequence of Corollary 3.5. Assume now $t = r-1$. By Corollary 3.6 then $\mu = 0$ for $\mathcal{A}(\mathcal{T})$ or, equivalently, $\mathcal{A}(\mathcal{T})$ is a set of latin rectangles. The completeness of $\mathcal{A}(\mathcal{T})$ implies $\mathcal{T}$ to be complete, i.e. any two elements of X are connected by a block from $L \cup T$. As $T_i \cap T_j = \emptyset$ for all i,j with $i \ne j$, one obtains that $I(X, L \cup T)$ is an $S(2,\{r,v\},rv)$. □

Notice that complete sets of mutually orthogonal latin rectangles have been constructed in Quattrocchi, Pellegrino [19] for all v not exceeding the smallest prime divisor of r.

For a transversal seminet $(X;L_0,L_1,\ldots,L_t;T_1,T_2,\ldots,T_v)$ and $M \subseteq \{1,2,\ldots,v\}$, $M \ne \emptyset$, let $t_M := |\bigcap_{i\in M} T_i|$, and define $d := \Sigma_{M\subseteq\{1,2,\ldots,v\},M\ne\emptyset} (-1)^{|M|-1} \binom{t_M}{2}$. Then

$$(1) \qquad d + \Sigma_{l\in L} \binom{|l|}{2} \le \binom{|X|}{2},$$

with equality if the transversal seminet is almost-André.

5.3. PROPOSITION. *Let* $\mathcal{T}$ *be a transversal* $(t+1,r)$*-seminet with* v *transversals. Let* $\mathcal{T}$ *be* n*-regular, and assume* $I(X,L\cup T)$ *to be an* $S(2,\{r,n\},rn)$. *Then* $t = \frac{(v-n)(r-1)}{n(n-1)} - 1$, *and there are exactly* $\frac{v}{n}$ *transversals through each* $x \in X$.

Proof. As the transversal seminet is almost-André, (1) is valid with equality. Since $|T_i \cap T_j| \le 1$ for $i \ne j$, one obtains $d = \Sigma_{1\le i\le v} \binom{|T_i|}{2} = v\binom{r}{2}$. With $|l| = n$ for all $l \in L$ and $|X| = rn$, (1) turns into

$$v\binom{r}{2} + (t+1)r\binom{n}{2} = \binom{rn}{2}.$$

This can easily be transformed into the claimed equality for t. Let α be the number of transversals through some $x \in X$. Then $v-1 = rn-1 = \alpha(r-1) + (t+1)(n-1)$ together with the equality just proved yields $\alpha = \frac{v}{n}$. □

A class of examples satisfying the assumptions of Proposition 5.3 can be obtained as follows. Let $(X;L_0,L_1,\ldots,L_r)$ be an affine plane of order r. For $k \ge 2$, consider the transversal seminet $\mathcal{T} = (X;L_0,L_1,\ldots,L_{r-k};T_1,T_2,\ldots,T_{rk})$ where $T_1,T_2,\ldots,T_{rk}$ are the lines contained in $\bigcup_{r-k<i\le r} L_i$. Then $\mathcal{T}$ satisfies the assumptions of Proposition 5.3, with $n = r$ and $v = rk$. In this situation, Corollary 3.6 yields the bound $t \le r-2$ (since $\mu = 1$) while the exact value is $t = r-k$.

However, there also exist transversal seminets for which $I(X,L\cup T)$ is a linear space $S(2,\{r,n\},rn)$ with $r \ne n$. In terms of [13] these transversal seminets correspond exactly to the *partially resolvable 2-partitions* PRP 2-$(n,r,v;t+1)$

with the additional property that $v = nr$. Together with Proposition 5.3, the results of [13] and [5] on these designs imply

5.4. PROPOSITION. *An* n*-regular transversal* $(t+1,r)$*-seminet with* $I(X,L\cup T)$ *an* $S(2,\{r,n\},rn)$ *exists*

a) in the case $n=2$, $r\geq 3$ *if and only if eihter* $t=0$, $r=3$ *or* $t=r-1$,

b) in the case $n=3$, $r=2$ *if and only if* $t=0$,

c) in the case $n=3$, $r=4$ *if and only if* $t=0$ *or* $t=3$.

Notice that all seminets in the above proposition are either complete sets of mutually orthogonal latin rectangles ($t=r-1$) or transversal designs ($t=0$).

6. SOME STRUCTURES RELATED TO TRANSVERSAL SEMINETS

It is well-known that nets are equivalent to sets of mutually orthogonal latin squares, to transversal designs (via duality), to orthogonal arrays, to optimal codes, etc. Proposition 2.1 is a generalization of the first of these equivalences. Next, the second equivalence is generalized to transversal seminets.

Each transversal seminet $\mathcal{T} = (X;L_0,L_1,\ldots,L_t;T_1,T_2,\ldots,T_v)$ is equivalent to a *transversal packing*, via the incidence structure $I(L,X)$. Each block $x\in X$ hits each of the *groups* L_i in exactly one treatment $l\in L_i$, two treatments from distinct groups L_i,L_j are joined by at most one block, and there is no such block if $L_i=L_j$. Moreover, there are additional *main treatments* T_i; each T_i partitions the treatments of $I(L,X)$ into disjoint blocks.

Suppose now X and each L_i to be linearly ordered, i.e. $X=\{x_1,x_2,\ldots\}$ and $L_i=\{l_i^1,l_i^2,\ldots,l_i^r\}$. Let the matrix $B=(b_{ij})$ of size $(t+1)\times|X|$ be defined by $b_{ij}=k :\Longleftrightarrow x_j\in l_{i-1}^k$. This matrix is an OA (*orthogonal array*) if the seminet is a net (see e.g. [9]). In the general case, the set of all $|X|$ columns of B forms a *code* of *length* $t+1$ over the alphabet $\{1,2,\ldots,r\}$ with $|X|$ *words* and *minimal distance* t, since any two distinct columns coincide in at most one position. Each transversal T_j corresponds to a family of codewords (columns of B) such that, for all $k\in\{1,2,\ldots,r\}$ and all $i\in\{0,1,\ldots,t\}$, there is exactly one codeword in this family with value k in row i. The transversal T_j can also be regarded as an *injective diagonal* subset of $\{1,2,\ldots,r\}^{t+1}$ (i.e. as a set of r words of length $t+1$ over $\{1,2,\ldots,r\}$ which differ in any coordinate; see [6]).

Let the associated permutation arrays $A_1,A_2,\ldots,A_t$ now be latin rectangles or, equivalently, let $T_i\cap T_j=\emptyset$ for all i,j, $i\neq j$. Assume the numbering of the points $x\in X$ and of the lines $l\in L_i$ $(i\geq 1)$ now be more special than above: Let

$l = l_i^k$ if $l \cap T_1 \subseteq l_o^k$, and let $x = x_j$ with $j = (i-1)r + k$ if $x \in T_i \cap l_o^k$. Under these assumptions the first row of B satisfies $b_{1j} = k$ if $j \equiv k \pmod r$ and, for $m>1$, the m th row of B becomes a "linearization" of the latin rectangle A_{m-1}, i.e. $(b_{m,(i-1)r+1}, b_{m,(i-1)r+2}, \dots, b_{m,ir})$ is the i th row of A_{m-1}. Notice that the transversal T_i corresponds now to the set of columns $(i-1)r+1$, $(i-1)r+2, \dots, ir$ of B. For example, considering the simple complete set $A_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ of mutually orthogonal latin rectangles, one obtains

$$B = \begin{pmatrix} 1 & 2 & 3 & 1 & 2 & 3 \\ 1 & 2 & 3 & 2 & 3 & 1 \\ 1 & 2 & 3 & 3 & 1 & 2 \end{pmatrix}.$$

The permutation arrays A and A' are similar exactly if, for all j, column j of A' can be obtained from column j of A by renaming the symbols. Let $Q(A,A')$ be the $r \times r$ matrix (q_{ij}) defined by $q_{ij} = k$ if the i's in column j of A become k's in column j of A', and $q_{ij} = *$ otherwise. For example,

$$Q\left(\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \right) = \begin{pmatrix} 1 & * & 2 \\ 3 & 2 & * \\ * & 1 & 3 \end{pmatrix}.$$

As all permutation arrays are assumed to be standardized, $q_{ii} = i$ for all i in $Q(A,A')$. In no column of this matrix a symbol apoears twice. Therefore $Q(A,A')$ can be considered as (the multiplication table of) a partial left-cancellative groupoid defined on $\{1,2,\dots,r\}$. The permutation arrays A,A' are orthogonal if and only if this groupoid also is right-cancellative, and $Q(A,A')$ then becomes a partial quasigroup. Moreover, $Q(A,A')$ is a complete quasigroup if and only if the permutation arrays are row-transitive. Let $A_1, A_2, \dots, A_t$ be similar permutation arrays. By Proposition 1.2 of [4], these arrays are mutually orthogonal exactly if $Q(A_1,A_2)$, $Q(A_1,A_3)$, ..., $Q(A_1,A_t)$ are mutually orthogonal partial quasigroups. Notice that the associated transversal seminet is n-regular if and only if each $Q(A_i,A_j)$, $i \neq j$, has the following properties: Each $i \in \{1,2,\dots,r\}$ appears exactly n times, and there are exactly n distinct symbols in each row and in each column.

There are many incidence structures built from nets. Examples are transversal geometries [6], d-dimensional nets [21], and extensions of dual affine planes [23]. It will be interesting to study similar structures with nets replaced by transversal seminets.

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