Annals of Discrete Mathematics 6 (1980) 57–76

SOME PROPERTIES OF PERFECT MATROID DESIGNS

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Structure of perfect matroid designs is studied. Various known results and some new results on them are given. A list of small possible parameters of perfect matroid designs of rank 4 is given. Some unsolved problems are suggested.

1. Statements of problems and results

A *design* $D=(X, \beta)$ is a pair, where X is a finite set and $\beta \subseteq P(X)$, the set of all subsets of X. Elements of X are called the *treatments*, while those of β are called the *blocks*.

We define a *matroid* by the "hyperplane axioms". A matroid is a design $M=(X, \beta)$ such that

(i) $H_1, H_2 \in \beta, \quad H_1 \neq H_2 \Rightarrow H_1 \not\subset H_2$.

(ii) $H_1, H_2 \in \beta$, $H_1 \neq H_2$, $x \in X$, $x \notin H_1 \cup H_2 \Rightarrow$ there exists unique $H_3 \in \beta$ such that $H_1 \cap H_2 \cup \{x\} \subseteq H_3$.

Blocks of a matroid are called *hyperplanes*. For various definitions and results connected with matroids, see [26]. Subsets of X, which are intersections of hyperplanes are called *flats* of a matroid. Each subset $Y \subseteq X$ has a well-defined rank. If F is a flat of rank i and $x \in X \setminus F$, then, there is a unique flat of rank $(i+1)$ which contains $F \cup \{x\}$. Rank of X is said to be the rank of matroid.

Let M be a matroid of rank r. We will denote by M_i the matroid $M_i=(X, \beta_i)$, where β_i is the set of all i-flats (i.e., flats of rank i) of M, $1 \leq i \leq r$. M_i is called $(r\text{-}i)$th *truncation* of M. We will use the usual geometric terminology and call 1-flats points, 2-flats lines, etc.

A *perfect matroid design* (PMD) is a matroid of rank r, such that, for any integer i, the cardinality of all i-flats is the same number α_i, $0 \leq i \leq r$. For simplicity we consider only *simple* PMDs, i.e., PMD for which $\alpha_0=0$ and $\alpha_1=1$. We do not lose much generality by this assumption (see Section 2). We will write $\alpha_2=l$, $\alpha_{r-1}=k$ and $\alpha_r=|X|=v$. The set $\alpha=\{\alpha_0=0,\ \alpha_1=1,\ \alpha_2=l,\ \alpha_3, \ldots, \alpha_{r-1}=k,$

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$\alpha_r = v\}$ will be called the set of parameters for a PMD. We define the set $L = \alpha \setminus \{v, k\}$. PMD were first studied by Murty [19] and Murty *et al.* [20]. See also [10] for a short summary of various results and problems connected with PMDs. The following classical designs are examples of PMDs:

(i) Let $P_m(n, q) = (X, \beta)$, $0 < m < n$. X = set of points of an n-dimensional projective space $\mathrm{PG}(n, q)$ over a finite field $\mathrm{GF}(q)$; β = set of all m-dimensional subspaces of $\mathrm{PG}(n, q)$. $P_m(n, q)$ is a PMD of rank $m+2$ with

$$\alpha_i = \frac{q^i - 1}{q-1}, \quad 0 \leqslant i \leqslant m+1 \quad \text{and} \quad \alpha_{m+2} = \frac{q^{n+1}-1}{q-1}.$$

(ii) Let $A_m(n, q) = (X, \beta)$, $0 < m < n$. X = set of points of an n-dimensional affine space $\mathrm{AG}(n, q)$ over a field $\mathrm{GF}(q)$; β = set of m-dimensional subspaces of $\mathrm{AG}(n, q)$. $A_m(n, q)$ is a PMD of rank $m+2$, with $\alpha_i = q^i$, $0 \leqslant i \leqslant m+1$ and $\alpha_{m+2} = q^n$.

(iii) $D = (X, \beta)$ be $S(t, k, v) = S_\lambda(t, k, v)$, i.e., a t-design, with $\lambda = 1$ (see Section 2 for definition of $S_\lambda(t, k, v)$). D is a PMD of rank $t+1$, with $\alpha_i = i$, $0 \leqslant i \leqslant t-1$, $\alpha_t = k$, $\alpha_{t+1} = v$.

(iv) A particular case of (iii) is the design $D = (X, \beta)$, $\beta = P_k(X)$, the set of k-subsets of X. D is a PMD of rank $k+1$, with $\alpha_i = i$, $0 \leqslant i \leqslant k$, $\alpha_{k+1} = v$. Such a PMD is called a *trivoid*. Clearly a trivoid is a $(v-k)$-truncation of a Boolean algebra.

(v) An affine-triple system ($\mathrm{ATS}(m)$) of order m is a PMD of rank 4 with $\alpha_0 = 0$, $\alpha_1 = 1$, $\alpha_2 = 3$, $\alpha_3 = 9$, $\alpha_4 = 3^m$, $m \geqslant 4$. Affine triple systems were first studied by Hall [13] These systems have a closed relationship with commutative Moufang loops. Zassenhauss [27] was first to construct such a loop of order 81, which gives an affine triple system of order 4. Hall [13] independently constructed the system of order 4 and proved that it is unique. Beneteau [1] has shown recently that ATS(5) is unique ATSs have also been studied by Young [25]. Kantor [8] gives an interesting review of various results on these systems. We note that parameters of these systems are the same as those $A_2(m, 3)$, however, these systems are not isomorphic to $A_2(m, 3)$.

So far the above five examples are the only known PMDs. In Section 6 we will give a list of possible set of small parameters of PMDs of rank 4. We note that any PMD whose parameters are the same as those of $P_m(n, q)$ $m \geqslant 2$, $n \geqslant 3$, or those of $A_m(n, q)$ over $\mathrm{GF}(q)$, $q > 3$, $m \geqslant 2$, $n \geqslant 3$ are necessarily isomorphic to $P_m(n, q)$ or $A_m(n, q)$. This follows from a result of Tierlink [24] (see also [3, 21, 23]). Similarly any PMD with parameters of $S(t, k, v)$ is indeed $S(t, k, v)$. In view of above five examples, it seems interesting to develop the language of PMD. We believe that there are PMDs different from the above examples. In this paper we study PMDs in general. We ask some interesting questions connected with them; we answer them partially. A few unsolved problems are suggested.

Let L be a set of integers, k and v be any given integers. A design $D = (X, \beta)$ is said to be an *$A(L, k, v)$-system* [8,9] if

(i) $|B_1 \cap B_2| \in L$ for all $B_1, B_2 \in \beta$, $B_1 \neq B_2$.
(ii) $|B| = k$ for all $B \in \beta$.

Any PMD is an $A(L, k, v)$ where $L = \alpha \backslash \{v, k\}$. Any $A(L, k, v)$-system $D = (x, \beta)$ is said to be *maximal* if $K \subseteq X$, $|K| = k$, $|K \cap B| \in L$ for all $B \in \beta$ implies that $K \in \beta$. Any PMD is said to be *maximal* if it is maximal as an $A(L, k, v)$. The following result [8, 9] gives an upper bound on the number of blocks in an $A(L, k, v)$ and characterizes PMDs with large v. In fact it shows that PMDs with large v are $A(L, k, v)$'s with a maximum number of blocks.

Theorem 1a. *There exists a polynomial $v_0(x)$ such that for any $A(L, k, v)$, with b blocks and $v > v_0(k)$*

$$b \leqslant \prod_{X \in L} \frac{v - X}{k - X},$$

equality holds if and only if $A(L, k, v)$ is a PMD.

Conjecture on maximality. *Every simple PMD is maximal.*

Theorem 1a implies that any PMD with $v > v_0(k)$ is maximal. In Section 3 we will prove

Theorem 3a. *Any simple PMD of rank $\geqslant 3$, with*

$$v \leqslant k^2/\alpha_{r-2}$$

is maximal.

Theorem 3b. *Any simple PMD of rank 4 is maximal, unless $l > 2$ and*

$$\frac{(k-l)^2}{l-2} \leqslant v - k \leqslant \frac{(k-l)^2}{l-2} - \frac{(k-l)^2}{2(l-2)(l-1)^2}[-l + (4l-3)^{1/2}(l-2)].$$

Corollary 3c. *Any PMD of rank 4, with $l \leqslant 3$ is maximal.*

We remark that the examples (iii), $S(t, k, v)$'s are maximal. This follows from the well-known Johnson–Schönheim upper bound for codes and packings. In fact, the above results imply that all known PMDs (i)–(v) are maximal.

A matroid M is said to be *complete* if given any flat F there exist two hyperplanes H_1 and H_2 such that $F = H_1 \cap H_2$. In Section 4 we study the problem of completeness of PMDs. We prove that completeness depends only on parameters.

Theorem 4c. *Any PMD $M=(X, \beta)$ with parameters $\alpha=\{\alpha_0, \alpha_1, \ldots, \alpha_r\}$ is complete if and only if numbers $c(i, r-1)>0$ for all $0 \leqslant i \leqslant r-1$.*

(See Theorem 4a below for the definition and value of $c(i, r-1)$.)

In the process of proving Theorem 4c, we also obtain some necessary conditions for the existence of PMDs.

Theorem 4a. *Let M be a PMD of rank r, $\alpha=\{\alpha_0, \alpha_1, \ldots, \alpha_{r-1}, \alpha_r\}$ is the set of parameters for M. Let F be any i-flat of M and H be any u-flat of M, $F \subseteq H$. Then the number of u-flats H_1 of M, such that $F=H \cap H_1$ is given by $c(i, u)$, where*

$$c(i, u)=\sum_{i \leqslant w \leqslant u} t(w, u, r) \sum_{\boldsymbol{a} \in S(i, w)}(-1)^h \prod_{g=1}^{h} t(a_g, a_{g+1}, u).$$

In the summation

$$S(i, w)=\{\boldsymbol{a}=(a_0, a_1, \ldots, a_h) \mid (i=a_0<a_1<a_2<\cdots<a_h)=w\}$$

is the set of all chains of integers, starting at i and ending at w, h is the length of chain $\boldsymbol{a}$ and $t(i, j, k)$ are defined in Section 2.

Corollary 4b. *A necessary condition for the existince of a PMD of rank r, with parameters $\alpha=\{\alpha_0, \alpha_1, \ldots, \alpha_r\}$ is that $c(i, u) \geqslant 0$ for all $0 \leqslant i<u \leqslant r$.*

The problem of completeness for t-designs has been completely studied by Gross [12]. He has proved

Theorem 1b. *Any $S(t, k, v)$ is not complete if and only if (t, k, v) is one of the following:*

(i) $(2, n+1, n^2+n+1)$;
(ii) $(3, 6, 22)$;
(iii) $(4, 7, 23)$;
(iv) $(t, t+1, 2t+3)$ *and* $t+3$ *is a prime number*;
(v) $(5, 8, 24)$;
(vi) $(3, 4, 8)$;
(vii) $(3, 12, 122)$ *if it exists*;
(viii) $(t, t+1, 2t+2)$ *and* $t+2$ *is a prime number.*

The expressions $c(i, u)$ described in Corollary 4b and Theorem 4c are messy and hence difficult to apply. A simple Gross type formula [12, Lemma 6] for general PMDs would be very much desirable. In Section 4 we will also show that all large PMDs are complete.

Theorem 4d. *There exists a polynomial $v_1(x)$ such that any PMD M with $v>v_1(k)$ is complete.*

The following result follows easily from the properties of vector-spaces and the fact that completeness depends only on parameters.

Theorem 1c.

(i) *$P_m(n, q)$ is complete if and only if $n \geq 2m+1$.*
(ii) *$A_m(n, q)$ is complete if and only if $n \geq 2m$.*
(iii) *Affine triple systems are always complete.*

Thus for all known PMDs the problem of completeness is solved. A coline of a PMD is always the intersection of two hyperplanes. In Section 4 we show that any PMD in which a coplane is not the intersection of two hyperplanes is an extension of a projective plane. We do not have any interesting observation for flats of smaller rank. For an interesting characterization of $P_{d-1}(d, q)$ or $A_{d-1}(d, q)$ as matroids which are extensions of symmetric BIBDs, see [28]. A PMD $M = (X, \beta)$ is said to be a *PMD-scheme* if β is a subscheme of a Johnson association scheme (see Section 5 for definitions). PMD-schemes are discussed briefly in Section 5.

Section 6 is devoted to the study of PMDs of rank 4, essentially from the point of view of existence. Existence of a PMD M implies existence of many BIBDs (see Section 2). Hence from the known conditions for BIBDs we get many necessary conditions on the parameters of a PMD. A list of small possible parameters of rank 4 PMD is given in Section 6. We also discuss some general results.

Finally we state the following result, which is an immediate consequence of Theorem 1a.

Theorem 1d.

(i) *If $D = (X, \beta)$ is PMD of rank r and with parameters $\alpha = \{\alpha_0, \alpha_1, \ldots, \alpha_{r-1} = k, \alpha_r = v\}$, then D is also BIBD (v, b, r, k, λ) such that*
(a) $b = \prod (v-\alpha_i)/(k-\alpha_i)$, $0 \leq i \leq r-2$.
(b) $|B_1 \cap B_2| \in \alpha$ for all $B_1, B_2 \in \beta$.

(ii) *There exists a polynomial $v_0(x)$ such that if D is any BIBD (v, b, r, k, λ) with $v > v_0(k)$ and satisfying conditions* (a) *and* (b) *for a given set α, then D is a PMD with parameters α.*

2. Preliminaries

We first define a $S_\lambda(t, k, v)$. A design $D(X, \beta)$ is called an $S_\lambda(t, k, v)$ (also called a t-design) if

(i) $|X| = v$;
(ii) $|B| = k$ for all $B \in \beta$;
(iii) Every t-subset of X is in exactly λ blocks.

$S_\lambda(t, k, v)$ with $t=2$, is also called BIBD (b, v, r, k, λ) where b is the number of blocks in D and r is the number of blocks containing any given $x \in X$. BIBDs and $S_\lambda(t, k, v)$s have been widely studied. see for example [20, 26] for various properties of BIBDs or $S_\lambda(t, k, v)$s.

Now let $M=(X, \beta)$ be a PMD of rank r and parameters L. Let F be any i-flat of M, K be an m-flat of M, $F \subseteq K$. Let us denote by $t(i, j, m)$ the number of j-flats of M containing F and which are contained in K. Then $t(i, j, m)$ is given by (see [20])

$$t(i, j, m) = \prod_{g=i}^{j-1} \frac{(\alpha_m - \alpha_g)}{(\alpha_j - \alpha_g)}. \tag{2.1}$$

The $(r-i)$th truncation M_i defined in Section 2 is a BIBD $(b_1, v_1, r_1, k_1, \lambda_1)$ with $v_1 = v$, $b_1 = t(0, i, r)$, $r_1 = t(1, i, r)$, $k_1 = t(0, 1, i)$, $\lambda_1 = t(2, i, r)$.

Given any matroid M one can define in a natural way a "projective" matroid (see [26]) $P(M) = (X_1, T)$. X_1 is the set of all 1-flats of M. T is the set of subsets $H \subseteq X_1$ such that $\bigcup_{h \in H} h$ is a hyperlane H_1 of M and $h \in H$ if and only if $h \subseteq H_1$. If M is a PMD of rank r and parameters α, then $P(M)$ is also PMD of rank r and parameters

$$\alpha' = \{\alpha'_i \mid 0 \leqslant i \leqslant r\}, \quad \text{where} \quad \alpha'_i = \frac{\alpha_i - \alpha_0}{\alpha_1 - \alpha_0}, \quad 1 \leqslant i \leqslant r$$

Let $M=(X, \beta)$ be any simple matroid. Let $K \subseteq X$, $M_K = (K, \beta_K)$, where $\beta_K = \{H \cap K \mid H \in \beta\}$ is a matroid, called the *induced matroid* on K. The rank of M_K is the rank of K in M. If M is PMD with parameters α and K is i-flat of M, then M_K is a PMD of rank i. Flats of M_K are flats of M contained in K.

Let M be any matroid. Let A be any $i-$flat of M. let $M_1^A = (X, \beta')$ be the design, where β' is the set of all hyperplanes of M containing A. Then M_1^A is a matroid. We will denote by M^A the matroid $P(M_1^A)$. If M is a PMD of rank r and parameters α, then M^A is a PMD of rank $r-1$ and parameters

$$\alpha' = \{\alpha'_l = (\alpha_{l+i} - \alpha_i)/(\alpha_{i+1} - \alpha_i) \mid 0 \leqslant l \leqslant r-1\}.$$

Thus the existance of a PMD implies the existance of many other PMDs and hence many BIBDs. This gives many necessary conditions on the parameters of PMDs. In Section 6 we use these conditions for the case of rank 4 PMDs. A matroid M of rank r will be called an *extension* of order i of matroid N of rank $r-1$ if $N \simeq M^F$ for some flat F of M of rank i.

3. Maximality

Proof of Theorem 3a. The technique used to prove the result is essentially the same as that in [22]. In fact the result is implicitly contained in the theorem proved in [22], however, we give a short proof here. Let us assume that $M=(x, \beta)$ is a

PMD of rank r, which is not maximal. Hence there is a set $K \subseteq X$, $|K| = k$, such that for all $H \in \beta$, $|K \cap H| = \alpha_i$, $0 \leq i < r-1$ but $K \notin \beta$. In particular K satisfies

$$|K \cap H| \leq \alpha_{r-2} \quad \text{for all } H \in \beta. \tag{3.1}$$

Now M is a BIBD (b, v, r', k, λ) with parameters

$$b = \frac{v(v-1)(v-\alpha_2)\cdots(v-\alpha_{r-2})}{k(k-1)(k-\alpha_2)\cdots(k-\alpha_{r-2})}, \tag{3.2}$$

$$\lambda = \frac{(v-\alpha_2)\cdots(v-\alpha_{r-2})}{(k-\alpha_2)\cdots(k-\alpha_{r-2})}, \tag{3.3}$$

$$r' = \frac{(v-1)\cdots(v-\alpha_{r-2})}{(k-1)\cdots(k-\alpha_{r-2})}. \tag{3.4}$$

We will show that the hypothesis on M implies $\alpha_{r-2}v > k^2$. Let $\beta = \{H_1, H_2, \ldots, H_b\}$ and let $x_i = |K \cap H_i|$, $i = 1, \ldots, b$. Using Fisher equations for K in BIBD M we have

$$\sum_{i=1}^{b} x_i = r'|K| = r'k \tag{3.5}$$

and

$$\sum_{i=1}^{b} x_i(x_i - 1) = \lambda|K|(|K|-1) = \lambda k(k-1). \tag{3.6}$$

Using (3.1) we have $x_i \leq \alpha_{r-2}$ for all i. Hence using (3.5) we get

$$k \cdot \frac{kb}{v} = k\frac{(v-1)(v-\alpha_2)\cdots(v-\alpha_{r-2})}{(k-1)(k-\alpha_2)\cdots(k-\alpha_{r-2})} \leq \alpha_{r-2}b$$

and so

$$\alpha_{r-2}v \geq k^2. \tag{3.7}$$

Now suppose $\alpha_{r-2}v = k^2$. Then we must have equality in (3.7) and hence $x_i = \alpha_{r-2}$ for all i. Using (3.6) and (3.4) we have

$$\alpha_{r-2}(\alpha_{r-2}-1)\frac{v(v-1)\cdots(v-\alpha_{r-2})}{k(k-1)\cdots(k-\alpha_{r-2})} = k(k-1)\frac{(v-\alpha_2)\cdots(v-\alpha_{r-2})}{(k-\alpha_2)\cdots(k-\alpha_{r-2})}. \tag{3.8}$$

Simplifying, using (3.8), we get

$$v + \alpha_{r-2} = 2k.$$

Hence

$$\frac{v}{k} - 1 = \frac{k-\alpha_{r-2}}{k} < \frac{k-\alpha_{r-2}}{\alpha_{r-2}} = \frac{k}{\alpha_{r-2}} - 1 \quad \text{or} \quad v - \alpha_{r-2} < k^2.$$

This contradicts the hypothesis and completes the proof of Theorem 3a.

Corollary 3.9 *The examples* (i), (ii) *and* (iv) *are maximal* (*see* [22]).

We now assume that $M=(X,\beta)$ is a PMD of rank 4, which is not maximal. Hence there is a set $K\subseteq X$, $K\notin\beta$, $|K|=k$, such that $|H\cap K|=\alpha_i$, $0\leqslant i\leqslant r-1$. If x and y are two points of M, we will denote by xy the line containing points x and y of M.

Lemma 3.10. $|K\cap xy|=l-(k-l)^2/(v-k)$ *for all* $x, y\in K$ (*note that* $l=\alpha_2$).

Proof. Let $x, y\in K$ and $|K\cap xy|=p$. Let H be any hyperplane of M, $x, y\in H$. Now $|H\cap K|\geqslant 2$, hence $|H\cap K|=l$. Also, if $z\in K$, $z\notin K\cap xy$, then clearly there is a unique hyperplane of M, containing x, y, z. Hence counting occurrences of such z in hyperplanes, we have

$$k-p=\lambda(l-p),$$

where $\lambda=(v-l)/(k-l)$ is the number of hyperplanes containing x and y. Thus

$$k-p=\frac{v-l}{k-l}(l-p).$$

Hence $p=l-((k-l)^2/(v-k))$.

Corollary 3.11. *If M is simple PMD of rank* 4, *which is not maximal, then*

(i) $(k-l)^2\equiv 0 \mod(v-k)$;
(ii) $(l-2)(v-k)\geqslant(k-l)^2$ *and* $l\neq 2$.

Proof. (i) is immediate from Lemma 3.10. (ii) follows from the fact that $p\geqslant 2$ and $k\neq l$.

Again consider the PMD M as described above. We can also assume $l\neq 2$ in view of the above corollary. From the proof of Lemma 3.10, it is clear that the induced matroid M_K is a PMD of rank 4, with parameters $\alpha'=\{0, 1, p, l, k\}$. Now let H_1 be a hyperplane of M_K. Consider the induced matroid $(M_K)_{H_1}$ of M_K. $(M_K)_{H_1}$ is clearly a BIBD with parameters $(l, l(l-1)/p(p-1), (l-1)/(p-1), p, 1)$. Now using Fisher's inequality we have $l-1\geqslant p(p-1)$. Substituting $p=l-((k-l)^2/(v-k))$, we get

$$l-1\geqslant\left(l-\frac{(k-l)^2}{v-k}\right)^2-l+\frac{(k-l)^2}{v-k}.$$

Simplifying we get

$$(l-1)^2(v-k)^2-(2l-1)(k-l)^2(v-k)+(k-l)^4\leqslant 0. \tag{3.12}$$

The left-hand side of above equation is a quadratic in $(v-k)$. The two roots of the quadratic are

$$\frac{1}{2(l-1)^2}[(2l-1)(k-l)^2 \pm ((2l-1)^2(k-l)^2 - 4(k-l)^4(l-1)^2)^{1/2}]$$
$$= \frac{(k-l)^2}{2(l-1)^2}[2l-1 \pm (4l-3)^{1/2}]$$
$$= \frac{(k-l)^2}{l-2} + \frac{1}{2(l-2)(l-1)^2}[-l \pm (4l-3)^{1/2}(l-2)].$$

Hence using corollary 3.11(ii) and (3.12) we have

$$\frac{(k-l)^2}{l-2} \leq v-k \leq \frac{(k-l)^2}{l-2} + \frac{(k-l)^2}{2(l-2)(l-1)^2}[-l+(4l-3)^{1/2}(l-2)].$$

Thus we have proved Theorem 3.6.

4. Completeness

Let M be a PMD of rank r, with parameters $\alpha = \{\alpha_0 = 0, \alpha_1 = 1, \alpha_2 = l, \ldots, \alpha_{r-1} = k, \alpha_r = v\}$. Let F be any i-flat of M. Let H be any u-flat of M, $F \subseteq H$. We will denote by $c(i, u)$ the number of u-flats H_1 of M, such that $H_1 \cap H = F$. We will show that $c(i, u)$ is independent of choice of F and H. Let us define for a given u function

$$\alpha_u(i, j) = t(i, j, u).$$

Clearly

$$\alpha_u(i, j) = 1, \quad 0 \leq i \leq u,$$
$$\alpha_u(i, j) = 0 \quad \text{unless } 0 \leq i \leq j \leq u.$$

Let $P_u = \{x \mid x \text{ is an integer } 0 \leq x \leq u\}$. P_u is a partially ordered set under the relation $\leq$. Let $I(P_u, R)$ denote the incidence algebra of partially ordered set P_u with coefficients in the field of real numbers (see [11, p. 81] or [2, p. 270]). The product in $I(P_u, R)$ is given by convolution $*$. $[f * g(i, j) = \sum_{i \leq u \leq j} f(i, u)g(u, i)]$. Now for given integers $0 \leq i \leq j$,

$$S(i, j) = \{\boldsymbol{a} = (a_0, a_1, \ldots, a_h) \mid a_0 = i < a_1 a_2 \cdots a_h = j\}$$

be the set of all chains of integers with starting point i and final point j.

For $\boldsymbol{a} = (a_0, a_1, \ldots, a_t) \in S(i, j)$ define

$$\alpha_u(\boldsymbol{a}) = (-1)^t \prod_{1 < i < t} \alpha_u(a_i, a_{i+1})$$

and

$$\beta_u(i, j) = \sum_{\boldsymbol{a} \in S(i, j)} \alpha_u(\boldsymbol{a}).$$

Clearly

$$\beta_u(i,i)=1,\quad 0\leqslant i\leqslant u$$
$$\beta_u(i,j)=0,\quad \text{unless } 0\leqslant i\leqslant j\leqslant u.$$

Thus $\beta_u \in I(P_u, R)$.

Lemma 4.1. *$\alpha_u * \beta_u(i,j)=\delta_{ij}$, $0\leqslant i\leqslant j\leqslant u$, i.e., α_u and β_u are inverses of each other in $I(P_u, R)$.*

Proof. $\alpha_u(i,i)=1$, $0\leqslant i\leqslant l$.

Hence α_u has an inverse in $I(P_u, R)$ (see [11, pp. 81–83] or [2]). α_u^{-1} is defined by

$$\alpha_u^{-1}(i,i)=\frac{1}{\alpha_u(i,i)}=1,$$
$$\alpha_u^{-1}(i,j)=-\sum_{i\leqslant w<j}\alpha_u^{-1}(i,w)\alpha_u(w,j). \tag{4.2}$$

We will show by induction on $j-i$, that $\alpha_u^{-1}=\beta_u$. Let i be given and

$$\alpha_u^{-1}(i,i)=1=\beta_u(i,i),\quad 0\leqslant i\leqslant u.$$

Suppose $\alpha_u^{-1}(i,j_1)=\beta_u(i,j_1)$ for all $0\leqslant i\leqslant j_1<j\leqslant u$. We will show that

$$\alpha_u^{-1}(i,j)=\beta_u(i,j).$$

Using the induction hypothesis and (4.2) we have

$$\alpha_u^{-1}(i,j)=-\sum_{i\leqslant w<j}\beta_u(i,w)\alpha_u(w,j), \tag{4.3}$$
$$\beta_u(i,w)=\sum_{\boldsymbol{a}\in S(i,w)}\alpha_u(\boldsymbol{a}).$$

Now for $i<j$

$$S(i,j)=\bigcup_{w=i}^{j-1}\{\boldsymbol{a}=(a_0,a_1,\ldots,a_{t-1},a_t)\mid a_t=j,(a_0,a_1,\ldots,a_{t-1})\in S(i,w)\}.$$

Hence clearly, the sum on the right-hand side of (4.3) can be written as

$$\sum_{\boldsymbol{a}\in S(i,j)}\alpha_u(\boldsymbol{a})=\beta_u(i,j).$$

This completes the proof.

Lemma 4.4. *$c(i,u)$ is independent of choice of F and H and is given by*

$$c(i,u)=t(i,u,r)-\sum_{i<w\leqslant u}t(i,w,u)c(w,u). \tag{4.5}$$

Proof. We prove by induction on $u-i$. The result is clearly true for $i=u$. Assume that the lemma is true for all F of rank $i+1$, $F \subseteq H$. Let the given F be of rank i. There are $t(i, w, u)$ flats $F_1 \subseteq H$, such that $F_1 \supseteq F$ and rank of F_1 is w. For each of these F_1 There exist $c(w, u)$ u-flats H_1 such that $H_1 \cap H = F_1$. There are $t(i, u, r)$ flats H_1 of rank u, in all, containing F. Hence we have

$$c(i, u) = t(i, u, r) - \sum_{i<w\leq u} t(i, w, u)c(w, u).$$

This completes the proof.

Proof of Theorem 4a. If $0 \leq i \leq j \leq u$ define $f(i, j) = c(i, j)$, $g(i, j) = t(i, j, r)$ and $g(i, j) = f(i, j) = 0$ otherwise. Then clearly $f, g \in I(P_u, R)$.

(4.5) may be written as

$$g(i, u) = \sum_{i\leq w\leq u} \alpha_u(i, w)f(w, u).$$

Hence using the fact that $\alpha_u * \beta_u = \delta_{ij}$ we have

$$f(i, u) = \sum_{i\leq w\leq u} \beta_u(i, w)g(w, u),$$

i.e.,

$$c(i, u) = \sum_{i\leq w\leq u} \beta_u(i, w)t(w, u, r). \tag{4.6}$$

Substituting values for $\beta_u(i, w)$, we get

$$c(i,u) = \sum_{i\leq w\leq u} t(w, u, r) \sum_{\boldsymbol{a}\in S(i, w)} (-1)^h \prod_{g=1}^{h} t(a_g, a_{g+1}, u)$$

where h is the length of chain $\boldsymbol{a}$. This completes the proof of Theorem 4a.

The Corollary 4b is immediate. Now a PMD M is complete if and only if $c(i, r-1) > 0$, hence we have also proved Theorem 4c.

The expressions for $c(i, u)$s given in Theorem 4a is messy, and hence difficult to apply.

The following trivial lemma is useful to check the completeness for PMD.

Lemma 4.7. *Let $M = (X, \beta)$ be a PMD of rank r. Let F be any i-flat of M. Then in M $H_1 \cap H_2 \neq F$ for all $H_1, H_2 \in \beta$ if and only if in M^F $H_1' \cap H_2' \neq \emptyset$ for all hyperplanes H_1' and H_2' of M^F.*

Corollary 4.8. *In a PMD M of rank r, $c(r-3, r-1) = 0$ if and only if M is an extension of order $r-2$ of a projective plane.*

Proof. Let F be any $r-3$ flat of M. M^F is clearly a BIBD with $\lambda = 1$. Using the lemma we have $|B_1 \cap B_2| = 1$ for all blocks B_1, B_2 of M^F. Hence M^F is a projective plane.

The following result of Gross [12] is immediate from the corollary and Cameron's characterizations [4] of Steiner systems which are extensions of projective plane.

Theorem 4.9. *In a Steiner system $S(t, k, u)$, $t>2$, $|B_1 \cap B_2| \neq t-2$ for any two blocks of $S(t, k, v)$ if and only if (t, k, v) is one of the following.*

(i) $(3, 6, 22)$,
(ii) $(4, 7, 23)$,
(iii) $(5, 8, 24)$,
(iv) $(3, 4, 8)$,
(v) $(3, 12, 112)$, *if it exists.*

We now give a proof of Theorem 4d.

Proof of Theorem 4d. Suppose $M=(X, \beta)$ is a PMD of rank r. Let F be any i-flat of M, $i<r-1$, such that $H_1 \cap H_2 \neq F$ for any $H_1, H_2 \in \beta$. We will show that $v \leqslant v_1(k)$ for some polynomial $v_1(x)$. Consider the PMD M^F of rank $r-i$. From the hypothesis, it is clear that M^F is $A(L_1, k_1, v_1)$ with

$$L_1=\left\{\beta_g=\frac{\alpha_g-\alpha_i}{\alpha_{i+1}-\alpha_i} \,\middle|\, i+1 \leqslant g \leqslant r-2\right\}, \qquad k_1=\frac{k-\alpha_i}{\alpha_{i+1}-\alpha_i}, \qquad v_1=\frac{v-\alpha_i}{\alpha_{i+1}-\alpha_i}.$$

Note that β_g is the size of (g − u)-flats in M^F. Now let b_1 = the number of blocks in M^F.

$$b_1=\prod_{t=i}^{r-2} \frac{(v_1-\beta_t)}{(k_1-\beta_t)}, \quad \text{where } \beta_i=0. \tag{4.10}$$

Also using Theorem 1a we can find a polynomial $v_1'(x)$ such that if $v>v_1'(k)$, then

$$b_1 \leqslant \prod_{g=i+1}^{r-2} \frac{(v_1-\beta_g)}{(k_1-\beta_g)}. \tag{4.11}$$

Using (4.10) and (4.11) we can find a polynomial $v_1''(x)$ such that $v \leqslant v_1''(k)$. Thus $v \leqslant v_1(k)$, where $v_1(x)=\max\{v_1'(x), v_1''(x)\}$. This completes the proof.

5. PMD-schemes

An *association scheme* with n classes on a set X (see, for example, [5, 6, 14]) is a partition of the set of 2-subsets of X into classes $T_1, \ldots, T_n$ such that

(i) the number a_i of $y \in X$ with $\{x, y\} \in T_i$ for given $x \in X$ depends only on i;
(ii) the number b_{ijk} of $z \in X$ with $\{x, z\} \in T_i$, $\{z, y\} \in T_j$ for given $\{x, y\} \in T_k$ depends only on i, j, k.

A *Johnson scheme* is an association scheme with k classes of all k-subsets of given v-set, such that $\{x, y\} \in T_i$ if and only if $|x \cap y| = k - i$. We will call a PMD $M = (X, \beta)$ with $\alpha_{r-1} = k$, $\alpha_r = v$ *PMD-scheme* if it is subscheme of Johnson scheme.

Any PMD of rank 3 is BIBD with $\lambda = 1$ and so it is a PMD-scheme (see [5]). From now on we consider only PMD of rank $r > 3$.

Any $P_i(d, q)$, $2 \leqslant i \leqslant d-1$, is PMD-scheme (This follows from [6, Theorems 4, 7] and from [14]).

Only $A_i(d, q)$, $3 \leqslant i \leqslant d-1$, which is a PMD-scheme is $A_{d-1}(d, q)$ (see [14, Theorem 5.2]).

Let $M = (X, \beta)$ be a Steiner system $S(t, k, v)$. Such a PMD will be a PMD-scheme if $|\{|H_1 \cap H_2| \mid H_1, H_2 \in \beta, H_1 \neq H_2\}| \leqslant \frac{1}{2}t + 1$ (see Cameron–Delsarte theorem [5]). Suppose now that $M = S(3, k, v)$ is PMD-scheme; then Cameron [5] proved that $v \leqslant 2 + \frac{1}{2}k(k-1)(k-2)$. Only known $S(3, k, v)$-scheme are Möbius (inversive) planes $I(q) = S(3, q+1 = k, q^2+1 = v)$ with 3 classes and $S(3, 4, 8)$, $S(3, 6, 22)$ and $S(3, 12, 112)$ (if it exists) with 2 classes. These 3 Steiner systems and also $S(4, 5, 11)$, $S(4, 7, 23)$, $S(5, 8, 24)$ are the only Steiner systems with $t > 2$ for which $|\{|H_1 \cap H_2| \mid H_1, H_2 \in \beta, H_1 \neq H_2\}| \leqslant \frac{1}{2}t + 1$ (see [12]) and so Cameron-Delsarte theorem [5] is applicable.

6. PMD of rank 4

Let M be a simple PMD of rank 4 with parameters $\alpha = \{0, 1, l, k, v\}$. Let $t = (k-1)/(l-1)$, $u = kt/l$, $\lambda = (v-l)/(k-l)$.

Let us consider the parameters of four following BIBD $(b_i, v_i, r_i, k_i, \lambda_i)$, $1 \leqslant i \leqslant 4$,

$$D_1 = S(2, l, k),$$
$$D_2 = S(2, (k-1)/(l-1), (v-1)/(l-1)),$$
$$D_3 = S(2, l, v),$$
$$D_4 = S_\lambda(2, k, v).$$

Lemma 6.1.

(1) $$b_1 = \frac{k(k-1)}{l(l-1)} = t^2 - \frac{t(t-1)}{l} = u,$$

$$(v_1, r_1, k_1, \lambda_1) = \left(k - tl - t + 1, \frac{k-1}{l-1} = t, l, 1\right);$$

(2) $$b_2 = \frac{(v-1)(v-l)}{(k-1)(k-l)} = \lambda^2 - \frac{\lambda(\lambda-1)}{t},$$

$$(v_2, r_2, k_2, \lambda_2) = \left(\frac{v-1}{l-1} = \lambda t - \lambda + 1, \frac{v-l}{k-l} = \lambda, \frac{k-1}{l-1} = t, 1\right);$$

(3) $$b_3 = \frac{v(v-1)}{l(l-1)} = ub_2 - (\lambda - 1)v_2$$

$$(v_3, r_3, k_3, \lambda_3) = (v, v_2, l, 1);$$

(4) $$b_4 = \frac{v(v-1)(v-l)}{k(k-1)(k-l)} = \lambda b_2 - \frac{\lambda(\lambda-1)v_2}{u},$$

$$(v_4, r_4, k_4, \lambda_4) = (v, b_2, k, \lambda).$$

In fact, the lemma follows from $k = tl - t + 1$, $v = \lambda k - \lambda l + l$ (from definitions of t, λ) and $r_i = \lambda_i(v_i - 1)/(k_i - 1)$, $b_i = v_i r_i / k_i$ (well-known properties of BIBD). For example,

$$b_3 = \frac{v}{l} r_3 = \frac{\lambda k - \lambda l + l}{l} r_3 = (1-\lambda)r_3 + r_3 \frac{k}{l}\lambda = (1-\lambda)r_3 + r_3 \frac{\lambda u}{t}$$

$$= (1-\lambda)r_3 + \frac{\lambda t - \lambda + 1}{t}\lambda u = \lambda^2 u - (\lambda - 1)r_3 - u\frac{\lambda(\lambda-1)}{t} = ub_2 - (\lambda-1)v_2.$$

$$b_4 = \frac{v}{k} r_4 = \frac{\lambda k - \lambda l + 1}{k} r_4 = \lambda r_4 - \frac{l(\lambda - 1)}{k} r_4 = \lambda r_4 - \frac{t(\lambda-1)}{u} r_4$$

$$= \lambda r_4 - \frac{(\lambda-1)(\lambda^2 t - \lambda^2 + \lambda)}{u} = \lambda r_4 - \frac{\lambda(\lambda-1)(\lambda t - \lambda + 1)}{u} = \lambda b_2 - \frac{\lambda(\lambda-1)v_2}{u}.$$

Let $(b_i, v_i, r_i, k_i, \lambda_i)$ be an ordered set of five positive integers such that $b_i = v_i r_i / k_i$, $r_i = \lambda(v_i - 1)/(k_i - 1)$, $r_i \geqslant k_i$; then we say that this *five-tuple* is *admissible.* The existence of a BIBD $(b_i, v_i, r_i, k_i, \lambda_i)$ will imply that $(b_i, v_i, r_i, k_i, \lambda_i)$ is an admissible five-tuple (from well-known divisibility conditions and Fisher inequality for BIBD).

Let us call *triple* of integers (l, k, v), $0 < l < k < v$, *admissible,* if all $(b_i, v_i, r_i, k_i, \lambda_i)$, $1 \leqslant i \leqslant 4$, are admissible five-tuples and if all numbers

$$\frac{k-1}{l-1}, \quad \frac{k(k-1)}{(l-1)}, \quad \frac{v-l}{k-l}, \quad \frac{(v-1)(v-l)}{(k-1)(k-l)} \quad \text{and} \quad \frac{v(v-1)(v-l)}{k(k-1)(k-1)}$$

are integers.

Lemma 6.2. *The triple (l, k, v) of integers, $0 < l < k < v$, is admissible if and only if*

$$t, \quad \lambda, \quad \frac{t(t-1)}{l}, \quad \frac{\lambda(\lambda-1)}{t}, \quad \frac{\lambda(\lambda-1)(\lambda t - \lambda + 1)}{u}$$

are integers and $l \leqslant t \leqslant \lambda$.

Proof. t, λ are integers if and only if $(l-1) | (k-l) | (v-k)$. Let t, λ be integers. $(b_1, v_1, r_1, k_1, \lambda_2)$ is admissible if and only if u is an integer, $l \leqslant t$, i.e., if and only if $t(t-1)/l$ is integer, $l \leqslant t$.

$(b_2, v_2, r_2, k_2, \lambda_2)$ is admissible if and only if $\lambda(\lambda-1)/t$ is integer, $t \leqslant \lambda$.

$(b_3, v_3, r_2, k_3, \lambda_3)$ is admissible if both $(b_i, v_i, r_i, k_i, \lambda_i)$, $i = 1, 2$ are admissible.

$(b_4, v_4, r_4, k_4, \lambda_4)$ is admissible if and only if $\lambda(\lambda-1)/t$, u, $\lambda(\lambda-1)(\lambda t-\lambda+1)/u$ are integers and $b_2 \geqslant k$.

$$b_2 \geqslant k = \lambda^2(t-1)+\lambda-kt \geqslant 0 \Leftrightarrow \lambda \geqslant \lambda'$$

where

$$\lambda' = -\frac{1}{2(t-1)} + \left(\frac{1}{4(t-1)^2}+\frac{kt}{t-1}\right)^{1/2}.$$

But

$$l \leqslant t \Leftrightarrow k \leqslant t^2-t+1 \Rightarrow \lambda' \leqslant -\frac{1}{2(t-1)} + \left(\frac{1}{2(t-1)}+\frac{t}{t-1}+t^2\right)^{1/2} = t.$$

Also $l = t$ if $b_2 = k$.

The numbers mentioned in the statement of the lemma are integers if and only if t, u, λ, b_2, b_4 are integers.

Let $0<l<k<v$ be given integers such that $t=(k-1)/(l-1) \leqslant 15$ is an integer. Below we give a list of possible PMDs for $l=2$, $t-1$, t and all possible parameters (l, k) for the remaining case $2<l<t-1$. For $l=2$ the PMD is $S(3, k, v)$. The existence of $S(2, k-1, v-1)$ is necessary for the existence of $S(3, k, v)$. $S(3, k, v)$ exists for $k=4$ if and only if $v \equiv 2, 4 \pmod 6$, $v \geqslant 8$ (Hanani [15]). It also exists for $k-1=q$ (a power of prime), $v=q^2+1$ (i.e., inversive plane IP(q)). Also there exists $S(3, 6, 22)$ and $S(3, 4, 10)$ related to Mathieu groups. Now for $l=t>2$ the PMD is $P_2(d, l-1=q)$ where $v=(q^{d+1}-1)/(q-1)$, it exists if and only if q is a power of a prime, since $d \geqslant 3$. For $l=t-1>2$ the PMD is $A_2(d, l)$ where $v=l^d$; it exists if and only if l is a power of prime, as $d \geqslant 3$.

For $2<l<t-1 \leqslant 14$ we give in Table 1 all possible (l, k) such that $l \mid k(k-1)$; for all of these (l, k), except $(l, k)^*$ BIBD $D_1 = S(2, l, k)$ is known to exist.

Table 1

$t=(k-1)/(l-1)$	$(l, k)=(2, t+1)$	$(l, k)=(t, t^2-t+1)$	$(l, k)=(t-1, (t-1)^2)$	All other possible (l, k)s
2	$S(3, 3, v)$	$S(3, 3, v)$	—	—
3	$S(3, 4, v)$	$P_2(d, 2)$	$S(3, 4, v)$	—
4	$S(3, 5, v)$	$P_2(d, 3)$	$A_2(\lg_3 v, 3)$, ATSs	—
5	$S(3, 6, v)$	$P_2(d, 4)$	$A_2(\lg_4 v, 4)$	—
6	$S(3, 7, v)$	$P_2(d, 5)$	$A_2(\lg_5 v, 5)$	(3, 13)
7	$S(3, 8, v)$	—	—	(3, 15)
8	$S(3, 9, v)$	$P_2(d, 7)$	$A_2(\lg_7 v, 7)$	(4, 25)
9	$S(3, 10, v)$	$P_2(d, 8)$	$A_2(\lg_8 v, 8)$	(3, 19), (4,28), (6, 46)*
10	$S(3, 11, v)$	$P_2(d, 9)$	$A_2(\lg_9 v, 9)$	(3, 21), (5, 41), (6, 51)*
11	$S(3, 12, v)$	—	—	(5, 45)
12	$S(3, 13, v)$	$P_2(d, 11)$	$A_2(\lg_{11} v, 11)$	(3, 25), (4, 37), (6, 61)*
13	$S(3, 14, v)$	—	—	(3, 27), (4, 40), (6, 66)*
14	$S(3, 15, v)$	$P_2(d, 13)$	$A_2(\lg_{13} v, 13)$	(7,85)*
15	$S(3, 16, v)$	—	—	(3.31), (5, 61), (6, 76)* (7, 91), (10, 136)*

— means "does not exist".

Lemma 6.3. *All admissible* (l, k, v) *for* $2<l<t-1\leqslant 8$ *are*;

(1)	$(l, k)=(3,13)$	$\lambda\equiv 0, 1, 13, 18, 27, 31$	(mod 39),
	$(t, u)=(6, 26)$	$v\equiv 3, 13, 133, 183, 273, 313$	(mod 390),
	$v=10\lambda+3>k$	$\dfrac{v-1}{l-1}\equiv 1, 6, 66, 91, 136, 156$	(mod 195);
(2)	$(l, k)=(3, 15)$	$\lambda\equiv 0, 1, 14, 15, 21, 29$	(mod 35),
	$(t, u)=(7, 35)$	$v\equiv 3, 15, 171, 183, 255, 351$	(mod 420),
	$v=12\lambda+3>k$	$\dfrac{v-1}{l-1}\equiv 1, 7, 85, 91, 127, 175$	(mod 210);
(3)	$(l, k)=(4, 25)$	$\lambda\equiv 0, 1, 25, 32, 57, 176$	(mod 200),
	$(t, u)=(8, 50)$	$v\equiv 4, 25, 529, 676, 1201, 3700$	(mod 4200),
	$v=21\lambda+4>k$	$\dfrac{v-1}{l-1}\equiv 1, 8, 176, 225, 400, 1233$	(mod 1400);
(4)	$(l, k)=(3, 19)$	$\lambda\equiv 0, 1, 19, 45, 64, 153$	(mod 171),
	$(t, u)=(9, 57)$	$v\equiv 3, 19, 307, 723, 1027, 2451$	(mod 2736),
	$v=16\lambda+3>k$	$\dfrac{v-1}{l-1}\equiv 1, 9, 153, 361, 513, 1225$	(mod 1368);
(5)	$(l, k)=(4, 28)$	$\lambda\equiv 0, 1, 27, 28, 36, 55$	(mod 63),
	$(t, u)=(9, 63)$	$v\equiv 4, 28, 652, 676, 868, 1324$	(mod 1512),
	$v=24\lambda+4>k$	$\dfrac{v-1}{l-1}\equiv 1, 9, 217, 225, 289, 441$	(mod 504);
(6)	$(l, k)=(6, 46)$	$\lambda\equiv 0, 1, 46, 135, 162, 181$	(mod 207),
	$(t, u)=(9, 69)$	$v\equiv 6, 46, 1846, 5406, 6786, 7246$	(mod 8280),
	$v=40\lambda+6>k$	$\dfrac{v-1}{l-1}\equiv 1, 9, 369, 1081, 1297, 1449$	(mod 1656).

Proof. We will use the conditions $t\mid(\lambda(\lambda-1)$, and $u\mid\lambda(\lambda-1)(\lambda t-\lambda+1)$ of Lemma 6.2 and following observations:

(a) if t is a power of prime, then $t\mid\lambda(\lambda-1)$ if and only if $\lambda\equiv 0, 1 \pmod t$;
(b) if $\frac{1}{2}t$ is a power of odd prime, then $t\mid\lambda(\lambda-1)$ if and only if $\lambda\equiv 0, 1 \pmod{\frac{1}{2}t}$.
Case $(l, k)=(3,13)$: $t\mid\lambda(\lambda-1)\Leftrightarrow\lambda\equiv 0, 1 \pmod 3$, because $\frac{1}{2}t=3$ is prime.

$$u\mid\lambda(\lambda-1)(t\lambda-\lambda+1)\Leftrightarrow 26\mid\lambda(\lambda-1)(5\lambda+1-26)\Leftrightarrow 2\cdot 13\mid 5\cdot\lambda(\lambda-1)(\lambda-5)$$
$$\Leftrightarrow 13\mid\lambda(\lambda-1)(\lambda-5),$$

i.e., $\lambda\equiv 0, 1, 5 \pmod{13}$. So

$$\lambda\equiv 0 \pmod 3\Leftrightarrow\lambda\equiv 0, 27, 18 \pmod{39},$$
$$\lambda\equiv 1 \pmod 3\Leftrightarrow\lambda\equiv 13, 1, 31 \pmod{39}.$$

Case $(l, k)=(3, 15)$: $t\mid\lambda(\lambda-1)\Leftrightarrow\lambda\equiv 0, 1 \pmod 7$, because $t=7$ is prime.

$$u\mid\lambda(\lambda-1)(t\lambda-\lambda+1)\Leftrightarrow 35\mid\lambda(\lambda-1)(6\lambda+1+35)\Leftrightarrow 35\mid\lambda(\lambda-1)(\lambda+6).$$

In the subcase $\lambda \equiv 0 \pmod 7$ we have $\lambda = 7x$,

$$35 \mid 7x(7x-1)(7x+6) \Leftrightarrow 5 \mid x(7x-1)(7x+6) \Leftrightarrow 5 \mid x(7x-1+15)(7x+6+15)$$
$$\Leftrightarrow 5 \mid 7\cdot 7x(x+2)(x+3) \Leftrightarrow 5 \mid x(x-3)(x-2),$$

i.e., $x \equiv 0, 2, 3 \pmod 5$ and $\lambda \equiv 0, 14, 21 \pmod{35}$.

In the subcase $\lambda \equiv 1 \pmod 7$ we have

$$\lambda = 7x+1,\ 35 \mid (7x+1)7x(7x+7) \Leftrightarrow 5 \mid x(x+1)(7x+1) \Leftrightarrow 5 \mid x(x+1-5)(7x+1-15)$$
$$\Leftrightarrow 5 \mid x(x-2)(x-4),$$

i.e., $x \equiv 0, 2, 4 \pmod 5$ and $\lambda = 1, 15, 29 \pmod{35}$.

The other cases are proved, using similar techniques.

Let (l, k) be integers $2 \geqslant l \leqslant t = (k-1)/(l-1)$ and let t, $t(t-1)/l$ be integers. Denote by $v_0 = (k-l)k + l$. Let $\alpha = \{0, 1, l, k, v_0\}$ be the parameters of a PMD M, which may or may not exist.

Proposition 6.4. (a) *(l, k, v_0) is admissible;*

(b) *for $l = 2$ PMD M exists if and only if $S(3, k, (k-1)^2+1)$ exists; so a PMD M exists (it is inversive plane IP (q)) if $k-1 = q$ is a power of a prime;*

(c) *for $l = t$, M must be $P_2(4, q)$;*

(d) *for $2 < l < t-1 \leqslant 8$, $v_0 = \min v$, such that (l, k, v) is admissible, except for the cases $(l, k) = (3, 15), (4, 28)$;*

(e) *for $2 < l < t - l \leqslant 7$, $v = v_0$ D_1, D_3 exists always, D_2 exists for $(l, k) = (3, 15)$, $(3, 19)$.*

In fact, (a) follows from Lemma 6.2. (b), (c) we can see from cases $l = 2, t-1, t$. (d), (c) follow from Lemma 6.3 and known lists of BIBDs.

Let $v_0' = (k-l)(k-1) + l$. It is easy to see that (l, k, v_0') is admissible if and only if $k \mid 2l(2l-1)$; thus for $l = 3, 4$ the only admissible (l, k, v_0') are $(3, 15, 171)$ and $(4, 28, 652)$. $171 = \min v$ such that $(3, 15, v)$ is admissible, $652 = \min v$ such that $(4, 28, v)$ is admissible, any (l, k, v) with $v_0' < v < v_0$ is not admissible.

Denote by $m(l, k)$ the number $\frac{1}{2}l.\text{c.m.}(t, u)$, if both $\frac{1}{2}t$, $\frac{1}{2}u$ are odd integers and the number $\text{l.c.m.}(t, u)$ otherwise.

Proposition 6.5. *For any $\lambda \equiv 0, 1 \pmod{m(l, k)}$, $v > k$, the triple (l, k, v) is admissible.*

Proof. Let at least one of the numbers $\frac{1}{2}t$, $\frac{1}{2}u$ be not an odd integer. We have $\lambda \equiv 0, 1$ (mod(l.c.m.(t, u))). Hence t, u both divide $\lambda(\lambda-1)$ and (l, k, v) is admissible by Lemma 6.2. Let both $\frac{1}{2}t, \frac{1}{2}u$ be odd integers. Then $\text{l.c.m.}\frac{1}{2}(t, u) - 1 = m(l, k) - 1$ is even. We have $\lambda \equiv 0, 1$ (mod(l.c.m.$\frac{1}{2}(t, u)$)). Hence $\lambda \equiv 0, 1$

$(\mathrm{mod}(\mathrm{l.c.m.}(\frac{1}{2}t, \frac{1}{2}u)))$: so either $\frac{1}{2}t, \frac{1}{2}u$ both divide odd λ (and then t, u both divide $\lambda(\lambda-1)$) or $\frac{1}{2}t, \frac{1}{2}u$ both divide odd $\lambda-1$ (and then t, u both divide $\lambda(\lambda-1)$). So (l, k, v) is admissible via Lemma 6.2.

We have for $2<1<t-1\leq 8$ only one possible case ($(l, k)=(3, 13)$) when both $\frac{1}{2}t, \frac{1}{2}u$ are odd integers. From Lemma 6.3 it is clear that for each (l, k), $2<l<t-1\leq 8$, all v's, such that (l, k, v) is admissible, have 6 possible values modulo $m(l, k)$. Perhaps, any v from admissible (l, k, v) for a given (l, k) also has 6 possible values modulo $m(l, k)$, including $0, 1, k$ and sometimes $\pm(k-1)$ (as it happens for $(l, k)=(3, 15)$, $(4, 28)$).

We collect a few simple properties of PMD of rank 4 with $\alpha=\{0, 1, l, k, v\}$, in the following:

Proposition 6.6. (i) *If a PMD of rank* 4, *M exists then BIBDs* D_1, D_2, D_3, D_4 *exist.*

(ii) D_1 *is a symmetric BIBD, if and only if M is* $P_2(d, q)$, $d\geq 3$ (*see* [21]).

(iii) D_2 *is a symmetric BIBD, if M is* $P_2(3, q)$ *or* $A_2(3, q)$ *or* $S(3, 6, 22)$. (*if* rank $M\geq 5$, D_2 *is a symmetric BIBD iff M is* $P_{d-1}(d, q)$ *or* $A_{d-1}(d, q)$, *see* [28].)

(iv) D_3 *is never symmetric BIBD.*

(v) D_4 *is a symmetric BIBD if and only if M is* $P_2(3, q)$. (*Dembowski and Wagner's Theorem* [7], *see also* [23].)

Using Theorem 4c and some messy computing we can give the following information on completeness of PMD $M=(X, \beta)$ of rank 4 with $\alpha=\{0, 1, l, k, v\}$.

Proposition 6.7. (i) *A point* $x\in X$ *such that* $H_1\cap H_2\neq\{x\}$ *for any* $H_1, H_2\in\beta$ *exists if and only if* $\lambda=t$ (*i.e., if and only if* D_2 *is a projective plane*).

(ii) $H_1\cap H_2\neq\emptyset$ *for all* $H_1, H_2\in\beta$ *if and only if*

$$g=(k-1)(k\lambda-k+l)+l\left(\lambda^2-\frac{\lambda(\lambda-1)}{t}\right)\left(\frac{v}{k}-l\right)=0;$$

in particular, for $\lambda=k$ (*i.e., for* $v=k^2-kl+l=v_0$) $g\geq 0$ *with* $g=0$ *if and only if* $M=P_2(4, q)$.

(iii) *Any M with parameters* (l, k, v) *given in Lemma* 6.3 *is complete and maximal if it exists.*

Here we have for $(l, k)=(2, k)$, $(3, 7)$, $(3, 9)$, $(4, 13)$, $(4,16)$ all possible v's such that a PMD with $\alpha=\{0, 1, l, k, v\}$ exists. For others (l, k) we give first few v's such that (l, k, v) is admissible and, moreover, BIBDs D_1, D_2 exist ([16]); By $\underline{v}$ we denote such v's from this list that the BIBD D_2 is known to exist (see, for example, [18]). So now we need, for example, $D_4=S_{18}(2, 13, 183)$, $S_{15}(2, 15, 183)$,

Table 2. PMD of rank 4 with small α

α_0,	α_1,	$\alpha_2 = l$;	$\alpha_3 = k$	Possible $\alpha_4 = v$
0	1	2	k	any v such that $S(3, k, v)$ exists
0	1	3	7	$2^n - 1$, $n \geqslant 4$ (truncated PG $(n-1, 2)$)
0	1	3	9	3^n, $n \geqslant 3$ (truncated AG $(n, 3)$ and ATSs)
0	1	3	13	$133, \underline{183}, 273, \underline{313}, 393, 403, \ldots, \underline{663}, \ldots$
0	1	3	15	$171, \underline{183}, 225, 351, \ldots, \underline{675}, 77, \underline{843}, \ldots$
0	1	3	19	$\underline{307}, 723, \underline{1027}, 2451, 2739, 2755, 3043, 3459, \ldots$
0	1	4	13	$\frac{1}{2}(3^n - 1)$, $n \geqslant 4$ (truncated PG$(n-1, 3)$)
0	1	4	16	4^n, $n \geqslant 3$ (truncated AG $(n, 4)$)
0	1	4	25	$529, 676, \underline{1201}, 3700, 4204, 4225, \ldots, \underline{8404}, \ldots$
0	1	4	28	$652, 676, 868, 1324, 1516, \underline{1540}, \underline{2164}, \underline{2188}. \ldots$

$S_{19}(2, 19, 307)$, $S_{31}(2, 13, 313)$ with only 0, 1, 3 as sizes of block-intersections for $(l, k, v) = (3, 13, 183)$, $(3, 15, 183)$, $(3, 19, 307)$, $(3, 13, 313)$.

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