	0	1	2	3	4	5	6
0	2	5	8	2	6	2	4
1	5	4	3	6	4	5	3
2	8	3	2	5	3	3	6
3	2	6	5	6	4	3	4
4	6	4	3	4	2	6	5
5	2	5	3	3	6	8	3
6	4	3	6	4	5	3	5

$p = 211$

	0	1	2	3	4	5	6
0	6	2	4	5	2	6	8
1	2	8	5	3	7	4	5
2	4	5	6	4	6	6	3
3	5	3	4	2	7	6	7
4	2	7	6	7	5	3	4
5	6	4	6	6	3	4	5
6	8	5	3	7	4	5	2

$p = 239$

	0	1	2	3	4	5	6
0	12	5	8	6	4	2	2
1	5	2	7	6	6	7	7
2	8	7	2	7	5	5	6
3	6	6	7	4	6	5	6
4	4	6	5	6	6	6	7
5	2	7	5	5	6	8	7
6	2	7	6	6	7	7	5

$p = 281$

References Cited: [988, 1812, 1964]

8 Finite Groups and Designs

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8.1 Introduction

8.1 A *group* $(P, \cdot)$ is a set P, together with a binary operation $\cdot$ on P, for which

1. if $x, y \in P$, then $x \cdot y \in P$,
2. an *identity* element $e \in P$ exists, i.e., $x \cdot e = e \cdot x = x$ for all $x \in P$;
3. $\cdot$ is *associative*, i.e. $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ for all $x, y, z \in P$;
4. every element $x \in P$ has an *inverse*, an element x^{-1} for which $x \cdot x^{-1} = x^{-1} \cdot x = e$.

8.2 If $G = (P, \cdot)$ is a group, $Q \subseteq P$, and $H = (Q, \cdot)$ is also a group, then H is a *subgroup* of G, written $H \leq G$. It is *proper* if H is not the trivial group $\{e\}$, nor the whole group G. The *index* $[G : H]$ of H in G is $|G|/|H|$.

8.3 If G is a group and $H \leq G$, a *right coset* of H in G is any subset of the form $Hx = \{hx : h \in H\}$ for a fixed element $x \in G$. A *left coset* of H in G is similarly defined to be any subset of G of the form $xH = \{xh : h \in H\}$ for $x \in G$ fixed.

8.4 **Remarks** All groups and sets are assumed here to be finite. As is customary, the symbol 1 denotes the rational integer 1, the identity element of a group, as well as the trivial subgroup $\{1\}$. The meaning intended is always clear from the context. If G is a group and $H \leq G$, two right cosets Hx, Hy are identical if and only if $xy^{-1} \in H$; otherwise, they are disjoint. Hence, the number of distinct right cosets of H in G is $[G : H]$. Similarly, the number of distinct left cosets of H in G is $[G : H]$.

8.5 A *permutation* π on the set X is a one to one and onto mapping $\pi : X \to X$. The set of all permutations on X, with composition of functions as a binary operation, forms a group, the *symmetric group* on X. It is denoted by $Sym(X)$ or if $X = \{1, 2, \ldots, n\}$, by $\mathcal{S}_n$. If G is a subgroup of $Sym(X)$, then G is a *permutation group.*

8.6 **Examples** of permutation groups of particular importance in design theory arise from considering a graph, design, or geometry. A permutation of the vertices, elements, or points which "preserves the structure" is an *automorphism*. For example, a permutation on the elements of a simple BIBD is an automorphism when it maps blocks to blocks, and sets that are not blocks to sets that are also not blocks. The collection of all automorphisms forms a permutation group, the *automorphism group* of the structure.

8.7 **Remarks** Automorphism groups of combinatorial structures form a rich source of permutation groups. Although most combinatorial objects admit only the trivial automorphism or have very small automorphism groups, objects with large automorphism groups are elegant and sought after. Designs, graphs, and geometries with large groups are often the easiest to construct, and the first to be discovered among members of specified families. Moreover, such objects can be presented efficiently, frequently in terms of generators of their automorphism groups and certain base blocks.

8.8 A subgroup N of a group G is a *normal* subgroup, written $N \lhd G$, if $a^{-1}Na \subseteq N$ for every $a \in G$.

8.9 A nontrivial group is *simple* if it has no proper, normal subgroups.

8.10 If $N \lhd G$, then, for each $x \in G$, the right coset Nx is identical with the left coset xN, and Nx is a *coset* of N in G. If $N \lhd G$, the collection G/N of all distinct cosets of N forms a group under the operation $Nx \cdot Ny = Nxy$, the *quotient* or *factor* group of G modulo N.

8.11 A *normal series* of a group G is a chain of subgroups

$$1 = G_n \lhd G_{n-1} \lhd \cdots \lhd G_1 \lhd G_0 = G$$

whose *factor groups* are G_i/G_{i+1} for $0 \leq i < n$.

8.12 A finite group is *solvable* when it has a normal series in which each factor group is cyclic of prime order.

8.13 The *socle* of a group G is the subgroup generated by the union of all *minimal normal* subgroups of G.

8.14 **Remarks** A number of references and computational tools were of considerable help in deriving structural information and completing Tables 8.62 and 8.66. The *Atlas* [598] was of particular utility, especially for the information on the primitive spectrum of those simple groups in the list which are included in the *Atlas*. An updated version of the *Atlas* was reprinted in 2003, and much of the data is presently maintained online at [2119]. Of utility also were the libraries of groups in the algebraic computational systems *Magma* [436], and *Gap* [1667], as well as standard references [700], [930], [1006], [2113]. To obtain and check the results, extensive computations were undertaken on a number of workstations, with the computational package *Pythia*, developed by Chouinard and Magliveras, as well as the computational algebra systems *Magma* and *Gap*.

8.2 Permutation Groups

8.15 A *group action* is a triple (X, G, exp), where X is a set, G a group, and $exp : X \times G \to X$ a mapping written as exponentiation, $((x, g) \mapsto x^g)$, such that:

1. $x^1 = x$ for any $x \in X$, where 1 is the identity of G, and
2. $(x^g)^h = x^{(gh)}$ for any $x \in X$, and any $g, h \in G$.

The notation (X, G, exp) is abbreviated to $G|X$. If $G|X$ is a group action, then $|X|$ is the *degree* of $G|X$.

8.16 **Example** (right regular action) Let G be any group, and let $X = G$. Define an action $G|X$ by $(x, g) \mapsto xg$. Now, (1) and (2) are satisfied since $x \cdot 1 = x$, and $(xg)h = x(gh)$.

8.17 **Example** (permutation groups) Let G be a permutation group, i.e. a subgroup of the symmetric group $\mathcal{S}_n$. If $x \in X$, and $g \in G$, let $x^g = g(x)$.

8.18 **Example** (right action on cosets) Let G be a group, $H \leq G$, and $\Omega = \{Hx_i : x_i \in G\}$ be the collection of all distinct right cosets of H in G. It is easy to check that the mapping $(Hx, g) \mapsto Hxg$ defines an action $G|\Omega$.

8.19 **Remark** The action in Example 8.16 is a special case of the action in Example 8.18, where H is the identity subgroup of G.

8.20 **Example** Let $G|X$ be a group action. Then G acts on the collection of all k-subsets of X in a natural way. For $S \subseteq X$ and $g \in G$, define S^g by $S^g = \{x^g : x \in S\}$. Some beautiful designs constructed in this way are given in §VI.24, where the symmetric group is acting on the set of all 2-subsets of a set of size n.

8.21 **Example** (G acts on G by conjugation) Let G be any group, and let $X = G$. Define $exp : X \times G \to X$ by $x^g = g^{-1}xg$.

8.22 Let $G|X$ be a group action and let $x \in X$. Then the *orbit of x under G* is $x^G = \{x^g : g \in G\}$. The orbits under G partition X. The *stabilizer of x in G* is $G_x = \{g \in G : x^g = x\}$.

8.23 **Proposition** G_x is a subgroup of G, and when G is finite, $|G| = |x^G| \cdot |G_x|$.

8.24 Let $G|X$ be a group action and let $g \in G$. The set of all elements of X *fixed* by g, $\{x \in X : x^g = x\}$, is denoted by $Fix(g)$. The cardinality of $Fix(g)$ is the *character* of g, denoted by $\chi_{G|X}(g)$ or simply $\chi(g)$, and the mapping $\chi : g \mapsto \chi(g)$ is the *character* of the group action.

8.25 The *kernel* of a given group action $G|X$, denoted by $ker(G|X)$, is the set $\{g \in G : Fix(g) = X\}$. Then $ker(G|X)$ is a normal subgroup of G. $G|X$ is *faithful* if and only if $|ker(G|X)| = 1$.

8.26 **Example** A permutation group is a faithful group action.

8.27 In a group G two elements x, y are *conjugate* if and only if $x = y^g = g^{-1}yg$ for some $g \in G$. Thus, two elements are conjugate if and only if they are in the same G-orbit in the action of Example 8.21. The G-orbits of G in the action of Example 8.21 are the *conjugacy classes* of G. G is the disjoint union of its conjugacy classes: $G = K_1 + K_2 + \cdots + K_c$ and c is the *class number* of G. Similarly, two subgroups $H_1, H_2 \leq G$ are *conjugate* in G (denoted by $H_1 \sim H_2$) if and only if $H_1 = H_2^g$ for some $g \in G$.

8.28 A group action $G|X$ is *transitive* if and only if X is a single G-orbit. Thus, $G|X$ is transitive if and only if for any two elements $x, y \in X$ there exists a group element $g \in G$ such that $x = y^g$.

8.29 **Remarks** If G is a group and $H \leq G$, the action described in Example 8.18 is transitive. Moreover, any transitive action $G|X$ is isomorphic to one of the type described in Example 8.18. If G is a group and $H_1, H_2 \leq G$, then the two actions of G on the two collections of right cosets of H_1, and H_2 respectively are isomorphic if and only if the two subgroups H_1 and H_2 are conjugate. These observations are fundamental for the determination of all inequivalent transitive permutation representations of a group.

8.30 If $G|X$ is a permutation group, G is $\frac{1}{2}$-*transitive* or *semi-regular* if all G-orbits of X are of length $|G|$. G is *regular* or *sharply transitive* on X if G is transitive and semi-regular.

8.31 G is k-*transitive* (or k-*fold transitive*) on X if the induced action of G on the collection of all ordered k-subsets of X is transitive. $G|X$ is k-*homogeneous*, or k^*-*transitive*, if G is transitive on the k-subsets of X.

8.32 G is *sharply* k-*transitive* (*sharply* k-*homogeneous*) on X if G is k-*transitive* (k-*homogeneous*) on X and the stabilizer in G of an ordered (unordered) k-subset of X is the identity subgroup.

8.33 **Remarks** $G|X$ is k-transitive if and only if $G_x|(X-\{x\})$ is $(k-1)$-transitive for each $x \in X$. If $G|X$ is k-transitive, then it is k-homogeneous. Surprisingly, the converse is also true when $k \geq 5$ [1470]. See Theorem 8.52.

8.34 A permutation group $G|X$ is of *type* k^* if it is k^*-transitive, but not k-transitive. If $G|X$ is a transitive permutation group, the *rank* of G is the number of orbits of G_x on X.

8.35 Let $G|X$ be a transitive group action. A subset Δ of X is a *block of imprimitivity* if for any $g \in G$, $\Delta \cap \Delta^g = \Delta$ or $\emptyset$. The subsets $\emptyset, X$ and the singleton sets $\{x\}, x \in X$ are always blocks of imprimitivity for $G|X$, the *trivial* blocks. If the only blocks of imprimitivity for $G|X$ are the trivial ones, then $G|X$ is a *primitive* action.

8.36 **Remarks** A combinatorial design having an automorphism group that is transitive on its points necessarily has isomorphic derived structures with respect to any two points. Thus transitive permutation groups are important to design theory. A transitive permutation group is either primitive or imprimitive. In Tables 8.62 and 8.63, the primitive groups of degree up to 50 are presented. Transitive imprimitive groups have primitive homomorphic images of degree dividing the degree of the given group; Proposition 8.37 shows how imprimitive groups can be reduced to primitive groups of smaller degree. Intransitive groups can be formed as subdirect products of transitive groups ([1006], p.63).

8.37 **Proposition** Let Δ be a nontrivial block of imprimitivity of the imprimitive group action $G|X$ and set $\Omega = \{\Delta_1, \ldots, \Delta_m\} = \Delta^G$, the orbit of Δ under G. Then

1. Ω is a partition of X, a *system of imprimitivity*.
2. If for each $g \in G$ one defines $\bar{g} : \Omega \mapsto \Omega$ by
$$\bar{g} = \begin{pmatrix} \Delta_1 & \Delta_2 & \cdots & \Delta_m \\ \Delta_1^g & \Delta_2^g & \cdots & \Delta_m^g \end{pmatrix},$$
then $\bar{G} = \{\bar{g} : g \in G\}$ forms a permutation group on Ω and $g \mapsto \bar{g}$ is a homomorphism of G onto $\bar{G}$,
3. $\bar{G}$ is transitive on Ω and is isomorphic to G/N where N is the set of permutations in G that fix each of $\Delta_1, \Delta_2, \ldots, \Delta_m$ as sets, i.e., the kernel of the homomorphism $g \mapsto \bar{g}$. N is the *kernel of imprimitivity* for this system of imprimitivity.

8.38 **Proposition** If $G|X$ is a transitive group action and $x, y \in X$, then the respective stabilizer subgroups G_x and G_y are conjugate in G. Moreover, $G|X$ is primitive if and only if G_x is a maximal subgroup of G.

8.39 For k a positive integer, G is $(k+\frac{1}{2})$-*transitive* if the stabilizer of each $x \in X$ is $(k-\frac{1}{2})$-transitive on $X-\{x\}$. G is $(k+1)$-*primitive* if G_x is k-primitive on $X-\{x\}$.

8.40 **Theorem** (Zassenhaus, 1934; Tits, 1952; Huppert, 1962) Let p be an odd prime, and m any odd positive integer. Then there is a unique sharply 3-transitive group of degree p^m+1, namely $\mathrm{PGL}_2(p^m)$. If m is even, then there are precisely two sharply

3-transitive groups of degree p^m+1. These groups are $\mathrm{PGL}_2(p^m)$ and $\mathrm{P\Delta L}_2(p^m)$. If $p=2$, and m is a positive integer, then $L_2(2^m)$ is the unique sharply 3-transitive group of degree $p^m+1=2^m+1$.

8.41 Remark (Feit, 1960; Ito, 1962; Suzuki, 1962) All doubly transitive groups which contain an element fixing exactly two points but no element except for the identity fixing more than two points have been determined. These all have degree p^m+1, or 2^p, where p is a prime.

8.42 Theorem (Huppert) Every solvable, primitive group of degree $1+p$, p a prime, is 2-transitive. Solvability is a necessary condition.

8.43 Theorem (Jordan) If G is a primitive solvable permutation group, then its degree is a prime power p^a, and it contains a normal, elementary abelian subgroup N of order p^a. If A is a transitive abelian subgroup of G, then $p>3$ implies that $A=N$. When $p=3$, A is elementary abelian.

8.44 Theorem (Jordan) If $G|X$ is k-transitive ($k\ge 2$) and N is a proper normal subgroup of G, then N is $(k-1)$-transitive, except when G is the symmetric group on X, or if $|X|$ is a power of 2 and $k=3$.

8.45 Theorem (Ito) Let G be $(k+\frac{1}{2})$-transitive on X, and let N be a nonregular normal subgroup of G, then N is k-transitive, except possibly when $k=1$ or 2.

8.46 Theorem (Wielandt and Huppert) If G is k-primitive, then a nonregular normal subgroup is k-transitive.

8.47 Theorem (Ito) For $k>2$, a nonregular normal subgroup of a k-transitive permutation group $G|X$ is $(k-1)$-primitive.

8.48 An *isomorphism* from group $(P,\cdot)$ to group $(Q,\circ)$ is a one-to-one correspondence $\phi: P\to Q$ for which $\phi(x\cdot y)=\phi(x)\circ\phi(y)$ for all $x,y\in P$. An *automorphism* is an isomorphism from a group onto itself. The automorphism is *inner* if it is conjugation ($x\mapsto a^{-1}xa$) by a fixed group element $a\in G$; otherwise, it is an *outer automorphism*. The collection of all automorphisms of a group G is a subgroup of $Sym(G)$, denoted by $Aut(G)$. The inner automorphisms of G form a normal subgroup $I(G)$ of $Aut(G)$. The quotient group $Aut(G)/I(G)$ is the *outer automorphism* group of G.

8.49 Theorem (Nagao) Let G be a 6-transitive permutation group of degree n. If the outer automorphism group of every simple subgroup of G is solvable, then G is $\mathcal{S}_n$ or $\mathcal{A}_n$.

8.50 Theorem The outer automorphism group of every finite simple group is solvable. (This was known as the Schreier Conjecture.)

8.51 Remark Theorem 8.50 has been proved using the classification theorem of finite simple groups. No proof has yet been found which does not use the classification theorem.

8.52 Theorem (O'Nan) If G is a nontrivial t-transitive group for which the composition factors of all its proper subgroups have solvable outer automorphism groups, then $t\le 5$. Hence, there do not exist 6-transitive groups, except for $\mathcal{S}_n$ and $\mathcal{A}_n$.

8.53 Theorem (Wagner) If $G|X$ is 3-transitive and $|X|>3$ is odd, then any nontrivial normal subgroup of G is also 3-transitive on X.

8.54 A permutation group $G|X$ is *set-transitive* if it is k-homogeneous for each $k\le|X|$.

8.55 Theorem (Beaumont and Peterson) Apart from $\mathcal{S}_n$ and $\mathcal{A}_n$, there are exactly four set-transitive groups

1. a metacyclic group of degree 5 and order 20,

2. a group $G \cong \mathcal{S}_5$ represented on six points,
3. $L_2(8)$ of order 504, represented on nine points, and
4. $\mathrm{P\Gamma L}_2(8) = Aut(L_2(8))$ of order 1512, represented on nine points.

8.56 Theorem (Livingstone and Wagner) If $G|X$ is a permutation group of degree n, and $2 < k \leq n/2$, then the number of G-orbits on unordered k-sets is at least the number of G-orbits on ordered $(k-1)$-sets. Thus if G is k^*-transitive, then it must be $(k-1)$-transitive. When $k \geq 5$, G must also be k-transitive.

8.57 Remark (Livingstone and Wagner) Permutation groups $G|X$ of type k^* exist only for $k = 2, 3$, and 4.

8.58 Theorem (Kantor) If $G|X$ is of type 4^*, and G_x is odd, then $G \cong L_2(5)$, $\mathrm{PGL}_2(5)$, $L_2(8)$, $\mathrm{P\Gamma L}_2(8)$, or $\mathrm{P\Gamma L}_2(32)$, in their usual permutation representations.

8.3 Groups and Group Operations

8.59 Table Notation for group operations and names for groups.

Symbol	Description
Operations on Groups:	
$A \times B$	The *direct product* of groups A and B.
A^n	The group $A^{n-1} \times A$, where A is a group, and n a positive integer.
$A \cdot B$	A *semidirect* product (*split extension*) of A by B (the action of B by conjugation on the normal subgroup A is not specified here).
$\mathbb{Z}_m \cdot_j \mathbb{Z}_n$	The semidirect product where the action of a generator of $\mathbb{Z}_n$ on $\mathbb{Z}_m$ is multiplication by $j \in \mathbb{Z}_m$.
$A \backslash B$	A general extension of A whose factor group by the normal subgroup A is B, but which is not a semidirect product of A by B.
$A \wr B$	The wreath product of A by B (which is isomorphic to some $A^n \cdot B$, $B \leq \mathcal{S}_n$).
$\mathcal{A}_n \cap H$	When H is a subgroup of $\mathcal{S}_n$ and contains odd permutations, the subgroup of index 2 in H consisting of the even permutations present.
Basic Groups:	
1	The trivial group.
$\mathbb{Z}_n$	The cyclic group of order n.
D_n	The dihedral group of order $2n$.
$\mathcal{S}_n$	The symmetric group on a set of n elements.
$\mathcal{A}_n$	The alternating group on a set of n elements.
V_q	The elementary abelian group of order q, q a prime power.
Q_8	The quaternion group of order 8.
Linear Groups (q a prime power):	
$\mathrm{GL}_n(q)$	The general linear group of all invertible matrices of dimension n over $\mathbb{F}_q$. This is equivalent to the group of $\mathbb{F}_q$-automorphisms of vector space $V = \mathbb{F}_q^n = \mathbb{F}_q \oplus \cdots \oplus \mathbb{F}_q$.
$\mathrm{AGL}_n(q)$	The group of *affine* transformations on V of the form $x \mapsto Ax + b$, where $A \in \mathrm{GL}_n(q)$ and $b \in \mathbb{F}_q^n$. Thus $\mathrm{AGL}_n(q) \cong \mathbb{F}_q^n \cdot \mathrm{GL}_n(q)$.
$F_{q,v}$	The subgroup of $\mathrm{AGL}_1(q)$ of order $q \cdot v$ of transformations of the form $x \mapsto Ax + b$ where $A^v = 1 \in \mathbb{F}_q$ and $b \in \mathbb{F}_q$. This is also known as the *Frobenius* group of order qv.
$\mathrm{SL}_n(q)$	The *special linear* group of dimension n over $\mathbb{F}_q$, the group of transformations of determinant 1 in $\mathrm{GL}_n(q)$.
$\mathrm{ASL}_n(q)$	The extension $\mathbb{F}_q^n \cdot \mathrm{SL}_n(q) < \mathrm{AGL}_n(q)$.
$L_n(q)$	(or $\mathrm{PSL}(n, \mathbb{F}_q)$ or $\mathrm{PSL}_n(q)$ or $\mathrm{LF}_n(q)$) The projective special linear group formed by taking $\mathrm{SL}_n(q)$ modulo its center.
$\mathrm{PGL}_n(q)$	$\mathrm{GL}_n(q)$ modulo its center.
$\Gamma\mathrm{L}_n(q)$	For $q = p^a$, p a prime, $a > 1$, the extension of $\mathrm{GL}_n(q)$ by the Galois group of $\mathbb{F}_q$ over $\mathbb{F}_p$.

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Symbol	Description
$\Sigma\mathrm{L}_n(q)$	For $q = p^a$, p a prime, $a > 1$, the extension of $\mathrm{SL}_n(q)$ by the Galois group of $\mathbb{F}_q$ over $\mathbb{F}_p$. $\mathrm{A\Gamma L}_n(q)$, $\mathrm{A\Sigma L}_n(q)$, $\mathrm{P\Gamma L}_n(q)$ and $\mathrm{P\Sigma L}_n(q)$ are similarly defined from these.
$\mathrm{P\Delta L}_2(q)$	If $p \neq 2$ and $q = p^2$, then $\mathrm{P\Gamma L}_2(q) \cong \mathrm{PSL}_2(q) \cdot V_4$ contains three extensions of the form $\mathrm{PSL}_2(q) \cdot \mathbb{Z}_2$. Two of these are $\mathrm{PGL}_2(q)$ and $\mathrm{P\Sigma L}_2(q)$. J.H. Conway suggested denoting the third extension of this sort by $\mathrm{P\Delta L}_2(q)$ (the *diagonal* of the other two).
$U_n(q)$	(or $\mathrm{PSU}_n(q)$, or $\mathrm{PSU}_n(q^2)$) The *projective special unitary* group, a subgroup of $L_n(q^2)$.
$S_{2n}(q)$	(or $\mathrm{PSp}_{2n}(q)$) The *projective symplectic* group, a subgroup of $L_{2n}(q)$.
Lie Type and Sporadic Simple Groups:	
$Sz(8)$, $Sz(32)$	The *Suzuki* groups of orders 29,120 and 32,537,600, respectively.
$G_2(3)$	The Chevalley group of order 4,245,696.
$R(2)'$	($\cong\ ^2F_4(2)'$) The *Ree-Tits* group of order 17,971,200.
M_n	for $n = 11, 12, 22, 23, 24$, the five well-known Mathieu simple groups.
J_a	The smallest Janko group, of order 175,600 with smallest transitive representation of degree 266.
HaJ	The *Janko-Hall* group of order 604,800 with minimal transitive representation on 100 points.
J_3	The Janko-Higman-McKay sporadic simple group of order 50,232,960.
HS	The *Higman-Sims* simple group of order 44,352,000.

8.4 Primitive Groups

8.60 Remarks Primitive permutation groups are the building blocks from which all permutation groups can be constructed and play an important role in group theory. Primitive permutation groups are particularly useful in constructing combinatorial designs and finite geometries. A classification of these groups, afforded by the O'Nan-Scott Theorem [1441], divides them into two types: those with *solvable socle* (the *affine groups*) and those with *non-solvable socle*. The primitive permutation groups with non-solvable socle of degree at most 1000 were determined in 1988 by Dixon and Mortimer [730].

The decade 1995-2005 has witnessed significant advances in computational group theory and specifically permutation group algorithms. During the same period, strong advances in computing hardware have made available considerable and inexpensive computing power to researchers. As a consequence, papers like [1801] and [770] have appeared. In [1801] the authors determine all primitive permutation groups of degree less than 2500, while in [770] the authors determine all solvable primitive permutation groups of degree at most 6560. This section presents only the primitive permutation groups of degree at most 50. There are two reasons for this choice: the first is that there is no space for additional data of this type in this volume, the second that the computational complexity of constructing combinatorial designs/geometries is exponential in $\binom{n}{k}/g$ where $n = v$ is the degree of the permutation group (also the number of points of the design saught), k the design block size, and g the group order.

8.61 Remark Table 8.62 contains information about the *primitive* groups of degree at most 50. Table 8.63 lists permutations which generate the groups in Table 8.62.

Table 8.62 has eight columns. The first column is an index of the form $d.i$, where d is the degree of the primitive permutation group and i is a serial index distinguishing primitive groups of the same degree. Column 2 contains the *name* of the group, which often also provides structural information about the group. Column 3 contains the *order* of the group. Column 4 provides information about the *depth* of transitivity of the group. For example, 3p! indicates that the group is *sharply 3-primitive*. The "p" notation indicates primitivity, the numeral 3 indicates the depth, (i.e., that the stabilizer of a point is 2-primitive on $d-1$ points), and the exclamation mark indicates that the stabilizer of two points is sharply primitive (hence also regular) on $d-2$ points. A notation such as 3t! means that the group is sharply 3-transitive. Similarly, 3t indicates that the group is 3-transitive, 3* that the group is 3-homogeneous, while 3*! indicates that the group is sharply 3-homogeneous. If the group is 1-primitive but not 2-primitive, then the stabilizer of a point is not primitive on $d-1$ points, and in these cases the orbit lengths for the action of the point-stabilizer are provided. For example, group 15.1 ($\mathcal{A}_6$ acting on 15 points) is just primitive but not 2-primitive. The stabilizer of a point, which happens to be isomorphic to $\mathcal{S}_4$, has 3 orbits on points, of lengths 1, 6, and 8 respectively. A superscript over an orbit length indicates the number of times that orbit length is encountered, so, for example, for group 16.1 column 4 contains $[1, 5^3]$, which means that the stabilizer of a point has 3 orbits of length 5, besides the fixed point.

Column 5 specifies generators of the group in the current primitive representation. When a generator is labeled in the form a_k, it is the *canonical* element of order k and degree d. This means that one starts writing cycles of length k, encompassing serially the points $1, 2, \cdots, d$ for as long as one can. If k does not divide d, then the last *res* points are left fixed, where *res* is the remainder of the division of d by k. For example, for group 10.8, $a_5 = (1,2,3,4,5)(6,7,8,9,10)$, while for group 15.5, $a_7 = (1,2,3,4,5,6,7)(8,9,10,11,12,13,14)(15)$. Other generators are listed in Table 8.63 under the relevant degree. The subscript of a generator indicates its order. An exponent over a specified generator, such as in a_{14}^2 for group 14.1, has the usual meaning. A generator listed in the form $ax+b$, simply indicates the element $x \mapsto ax+b$ over the Galois field $\mathbb{F}_p$ where $d = p$. Every primitive group listed has been given as a two-generator group with the exception of group 25.11 which is not a 2-generator group.

Column 6 describes the structure of the stabilizer of a point for the specific group. If group G is at least 2-primitive on X, the point stabilizer G_x, $x \in X$, is primitive on $X - \{x\}$ and can be described by referring to an entry in Table 8.62. This is the case, for example, for all primitive groups of degree 12. When the stabilizer of a point G_x is not primitive on the remaining points $X - \{x\}$, it is sometimes possible to describe G_x as being isomorphic (as an abstract group) to an earlier group in the table. For example, the group $G = M_{21} \cong L_3(4)$, of index 21.4, is doubly transitive, with point stabilizer G_x which is imprimitive on 20 points. Here, a nontrivial system of imprimitivity consists of a set of 5 blocks, each of size 4, and G_x is a split extension of the kernel of imprimitivity V_{16}, by the image of imprimitivity $\mathcal{A}_5$. If G_x is represented on the cosets of (maximal) $\mathcal{A}_5$, then G_x is a primitive group of degree 16, and is isomorphic to the group 16.11. Similar is the meaning of symbol $\approx$ in column 6 for groups 21.5, 21.6, and 21.7.

Column 7 describes the lattice of the groups appearing as primitive groups for a specific degree, under *conjugate subgroup inclusion*. In most cases groups are represented so that any conjugate subgroup inclusion is in fact a *subgroup inclusion*. The indices that appear in this column refer to groups under the same degree. For example, group 37.1 is contained in groups 37.2 and 37.3. Group 37.2 in groups 37.4 and 37.5, and so on. In a few cases, a specific conjugate inclusion is given, as for group

16.12.

In the eighth column, examples of combinatorial structures are given whose full automorphism group is the group listed in the second column of the same row. A *strongly regular graph* with parameters v, k, λ, μ is denoted by $\mathrm{srg}(v, k, \lambda, \mu)$. A t-(v, k, λ) design is denoted by $S_\lambda(t, k, v)$. All t-(v, k, λ) designs listed are simple. $PG_n(q)$ denotes the *projective space* of (projective) dimension n over the field of q elements. This list of combinatorial structures is by no means complete!

Table 8.63 displays generators for the groups of Table 8.62. The amount of data needed to specify group generators is minimized, by listing as few permutations as possible in Table 8.63 for each specific degree, and by generating various groups by powers of elements listed in Table 8.63, or powers of canonical elements. This provides economy in representation of the data. For example, although there are 22 primitive groups of degree 16, only six permutations of degree 16 need be listed in Table 8.63.

paginate later **8.62 Table** Primitive groups of degree at most 50.

Degree $n.i$	Group G	Order $\lvert G\rvert$	Trans/Rank	Generators	G_α	min H, $G < H \leq \mathcal{S}_n$	Combinatorial Structures
5.1	$\mathbb{Z}_5$	5	[1^5]	$a_5 : x+1$	1	2	
.2	D_5	10	[1,2^2]	$x+1, -x$	$\mathbb{Z}_2$	3, 4	
.3	$\mathrm{AGL}_1(5)$	20	2t!	$x+1, 2x$	$\mathbb{Z}_4$	5	
.4	$\mathcal{A}_5$	60	3p!	a_3, a_5	$\mathcal{A}_4$	5	
.5	$\mathcal{S}_5$	120	5p!	a_4, a_5	$\mathcal{S}_4$		
6.1	$L_2(5) \cong \mathcal{A}_5$	60	2p	a_5, b_3	5.2	2, 3	
.2	$\mathrm{PGL}_2(5) \cong \mathcal{S}_5$	120	3t!	a_5, b_6	5.3	4	
.3	$\mathcal{A}_6$	360	4p!	a_3, a_5	5.4	4	
.4	$\mathcal{S}_6$	720	6p!	a_5, a_6	5.5		
7.1	$\mathbb{Z}_7$	7	[1^7]	$a_7 : x+1$	1	2, 3	
.2	D_7	14	[1,2^3]	$x+1, -x$	$\mathbb{Z}_2$	4	
.3	$F_{7,3}$	21	2*!	$x+1, 2x$	$\mathbb{Z}_3$	4, 5	
.4	$\mathrm{AGL}_1(7)$	42	2t!	$x+1, 3x$	$\mathbb{Z}_6$	7	
.5	$L_3(2)$	168	2t	a_7, b_7	$\mathcal{S}_4$	6	$S_1(2,3,7)$
.6	$\mathcal{A}_7$	2520	5p!	a_3, a_7	6.3	7	
.7	$\mathcal{S}_7$	5040	7p!	a_6, a_7	6.4		
8.1	$\mathrm{AGL}_1(8)$	56	2p!	a_7, b_7	7.1	2	
.2	$V_8 \cdot F_{7,3}$	168	3*	a_7, b_6	7.3	5	
.3	$L_2(7) \cong L_3(2)$	168	3*	a_7, c_7	7.3	4, 6	
.4	$\mathrm{PGL}_2(7)$	336	3t!	a_7, b_8	7.4	7	
.5	$\mathrm{AGL}_3(2)$	1344	3t	a_7, d_7	7.5	7	$S_1(3,4,8)$
.6	$\mathcal{A}_8$	20160	6p!	a_4, a_7	7.6	7	
.7	$\mathcal{S}_8$	40320	8p!	a_7, a_8	7.7		
9.1	$F_{9,4}$	36	[1,4^2]	d_4, e_4	$\mathbb{Z}_4$	2,3,4	
.2	$V_9 \cdot D_4$	72	[1,4^2]	f_6, g_6	D_4	5	
.3	$\mathrm{AGL}_1(9)$	72	2t!	b_3, c_8	$\mathbb{Z}_8$	5	
.4	$V_9 \cdot Q_8$	72	2t!	b_4, c_4	Q_8	5, 6	
.5	$V_9 \cdot (\mathbb{Z}_8 \cdot_3 \mathbb{Z}_2)$	144	2t	c_8, e_6	$\mathbb{Z}_8 \cdot_3 \mathbb{Z}_2$	7	
.6	$V_9 \cdot (Q_8 \cdot \mathbb{Z}_3)$	216	2t	c_6, d_6	$Q_8 \cdot \mathbb{Z}_3$	7, 10	
.7	$\mathrm{AGL}_2(3)$	432	2t	a_8, b_8	$\mathrm{GL}_2(3)$	11	$S_1(2,3,9)$
.8	$L_2(8)$	504	4*	a_9, b_9	8.1	9	
.9	$\mathrm{P\Gamma L}_2(8)$	1512	4*	a_9, b_6	8.2	10	
.10	$\mathcal{A}_9$	9!/2	7p!	a_4, a_9	8.6	11	
.11	$\mathcal{S}_9$	9!	9p!	a_8, a_9	8.7		

Degree $n.i$	Group G	Order $\lvert G\rvert$	Trans/Rank	Generators	G_α	min H, $G < H \le \mathcal{S}_n$	Combinatorial Structures
10.1	$\mathcal{A}_5$	60	[1,3,6]	b_3, d_5	$\mathcal{S}_3$	2, 3	$S_2(2,3,10)$
.2	$\mathcal{S}_5$	120	[1,3,6]	b_6, d_5	D_6	4	srg$(10,3,0,1)$
.3	$L_2(9) \cong \mathcal{A}_6$	360	2p	b_5, c_5	9.1	4, 5, 6	
.4	$\mathrm{P\Sigma L}_2(9) \cong \mathcal{S}_6$	720	2p	b_6, c_6	9.2	7	$S_2(2,4,10)$
.5	$\mathrm{P\Delta L}_2(9) \cong M_{10}$	720	3t!	c_8, d_8	9.4	7	$S_3(3,5,10)$
.6	$\mathrm{PGL}_2(9)$	720	3t!	a_{10}, b_{10}	9.3	7	
.7	$\mathrm{P\Gamma L}_2(9)$	1440	3t	a_{10}, b_8	9.5	9	$S_1(3,4,10)$
.8	$\mathcal{A}_{10}$	10!/2	8p!	a_5, a_9	9.10	9	
.9	$\mathcal{S}_{10}$	10!	10p!	a_9, a_{10}	9.11		
11.1	$\mathbb{Z}_{11}$	11	$[1^{11}]$	$a_{11} : x+1$	1	2, 3	
.2	$F_{11,2}$	22	$[1,2^5]$	$x+1,\ -x$	$\mathbb{Z}_2$	4	
.3	$F_{11,5}$	55	2*!	$x+1,\ 4x$	$\mathbb{Z}_5$	4, 5	
.4	$\mathrm{AGL}_1(11)$	110	2t	$x+1,\ 2x$	$\mathbb{Z}_{10}$	8	
.5	$L_2(11)$	660	2p	a_{11}, b_5	10.1	6	$S_2(2,5,11)$
.6	M_{11}	7920	4t!	a_{11}, b_{11}	10.5	7	$S_1(4,5,11)$
.7	$\mathcal{A}_{11}$	11!/2	9p!	a_5, a_{11}	10.8	8	
.8	$\mathcal{S}_{11}$	11!	11p!	a_{10}, a_{11}	10.9		
12.1	$L_2(11)$	660	3*	a_{11}, d_{11}	11.3	2,3	
.2	$\mathrm{PGL}_2(11)$	1320	3t!	a_{12}, b_{12}	11.4	6	
.3	M_{11}	7920	3p	a_{11}, c_{11}	11.5	4	
.4	M_{12}	95040	5t!	a_{11}, b_{11}	11.6	5	$S_1(5,6,12)$
.5	$\mathcal{A}_{12}$	12!/2	10p!	a_6, a_{11}	11.7	6	
.6	$\mathcal{S}_{12}$	12!	12p!	a_{11}, a_{12}	11.8		
13.1	$\mathbb{Z}_{13}$	13	$[1^{13}]$	$a_{13} : x+1$	1	2, 3	$S_5(2,6,13)$
.2	$F_{13,2}$	26	$[1,2^6]$	$x+1, -x$	$\mathbb{Z}_2$	4, 5	$S_2(3,4,13)$
.3	$F_{13,3}$	39	$[1,3^4]$	$x+1, 3x$	$\mathbb{Z}_3$	5, 7	$S_1(2,3,13)$
.4	$F_{13,4}$	52	$[1,4^3]$	$x+1, 8x$	$\mathbb{Z}_4$	6	$S_2(3,4,13)$
.5	$F_{13,6}$	78	$[1,6^2]$	$x+1, 4x$	$\mathbb{Z}_6$	6, 8	
.6	$\mathrm{AGL}_1(13)$	156	2t!	$x+1, 2x$	$\mathbb{Z}_{12}$	9	$S_2(2,3,13)$,$S_5(2,5,13)$
.7	$L_3(3)$	5616	2t	a_{13}, b_{13}	≈ 9.7	8	$\mathrm{PG}_2(3)$
.8	$\mathcal{A}_{13}$	13!/2	11p!	a_6, a_{13}	12.5	9	
.9	$\mathcal{S}_{13}$	13!	13p!	a_{12}, a_{13}	12.6		
14.1	$L_2(13)$	1092	2p	a_{14}^2, b_{13}	13.5	2, 3	
.2	$\mathrm{PGL}_2(13)$	2184	3t!	a_{14}, b_{13}	13.6	4	$S_2(3,4,14)$

Degree $n.i$	Group G	Order $\lvert G\rvert$	Trans/Rank	Generators	G_α	min H, $G < H \le \mathcal{S}_n$	Combinatorial Structures
14.3	$\mathcal{A}_{14}$	14!/2	12p!	a_7, a_{13}	13.8	4	
.4	$\mathcal{S}_{14}$	14!	14p!	a_{13}, a_{14}	13.9		
15.1	$\mathcal{A}_6$	360	[1,6,8]	a_{15}^3, b_2	$\mathcal{S}_4$	2, 3	
.2	$\mathcal{S}_6$	720	[1,6,8]	a_{15}^3, c_2	$\mathcal{S}_4 \times \mathbb{Z}_2$	4	
.3	$\mathcal{A}_7$	2520	2t	a_{15}^3, b_7	$L_3(2)$	4	
.4	$L_4(2) \cong \mathcal{A}_8$	20160	2t	a_{15}, c_2	$\mathrm{AGL}_3(2)$	5	$\mathrm{PG}_3(2)$
.5	$\mathcal{A}_{15}$	15!/2	13p!	a_7, a_{15}	14.3	6	
.6	$\mathcal{S}_{15}$	15!	15p!	a_{14}, a_{15}	14.4		
16.1	$V_{16} \cdot \mathbb{Z}_5$	80	[1,5^3]	a_{16}^8, b_{15}^3	$\mathbb{Z}_5$	2, 3	$S_1(3,4,16)$
.2	$V_{16} \cdot D_5$	160	[1,5^3]	a_{16}^4, b_{15}^3	D_5	5, 6, 9, 11	
.3	$\mathrm{AGL}_1(16)$	240	2t!	a_{16}^8, b_{15}	$\mathbb{Z}_{15}$	6	
.4	$V_{16} \cdot (S_3 \times \mathbb{Z}_3)$	288	[1,6,9]	b_6^2, c_6	$S_3 \times \mathbb{Z}_3$	8, 15	
.5	$V_{16} \cdot F_{5,4}$	320	[1,5,10]	a_{16}^2, b_{15}^3	$F_{5,4}$	10, 13, 14	
.6	$V_{16} \cdot (D_5 \times \mathbb{Z}_3)$	480	2t	a_{16}^4, b_{15}	$D_5 \times \mathbb{Z}_3$	10, 15	
.7	$V_{16} \cdot (F_{9,4})$	576	[1,6,9]	b_4, b_6^2	9.1	12, 17	
.8	$V_{16} \cdot (\mathcal{S}_3 \times \mathcal{S}_3)$	576	[1,6,9]	b_4^2, c_6	$S_3 \times S_3$	12, 16	
.9	$V_{16} \cdot \mathcal{A}_5$	960	[1,5,10]	a_{16}^4, b_3	10.1	13, 17	
.10	$V_{16} \cdot G_\alpha$	960	2t	a_{16}^2, b_{15}	$A_8 \cap (F_{5,4} \times \mathcal{S}_3)$	16	
.11	$V_{16} \cdot \mathcal{A}_5$	960	2t	a_{16}^4, b_6^2	$\mathcal{A}_5$	14, 15, 17	
.12	$S_4 \wr S_2$	1152	[1,6,9]	b_4, c_6	9.2	$18^{b_4 b_3}$	
.13	$V_{16} \cdot \mathcal{S}_5$	1920	[1,5,10]	a_{16}^2, b_3	$\mathcal{S}_5$	18	$\mathrm{srg}(16,5,0,2)$
.14	$V_{16} \cdot \mathcal{S}_5$	1920	2t	a_{16}^2, b_6	10.2	16, 18, 19	
.15	$\mathrm{AGL}_2(4)$	2880	2t	b_6^2, b_{15}	$\mathcal{A}_5 \times \mathbb{Z}_3$	16	
.16	$\mathrm{A\Gamma L}_2(4)$	5760	2t	b_6, b_{15}	$A_8 \cap (\mathcal{S}_5 \times \mathcal{S}_3)$	20	$S_1(2,4,16)$
.17	$V_{16} \cdot \mathcal{A}_6$	5760	2p	b_3, b_6^2	15.1	18, $19^{b_{15}}$	
.18	$V_{16} \cdot \mathcal{S}_6$	11520	2p	b_3, b_6	15.2	20	
.19	$V_{16} \cdot \mathcal{A}_7$	40320	3t	a_{16}^2, b_7	15.3	20	
.20	$\mathrm{AGL}_4(2)$	322560	3t	b_3, b_{15}	15.4	21	$S_1(3,4,16)$
.21	$\mathcal{A}_{16}$	16!/2	14p!	a_8, a_{15}	15.5	22	
.22	$\mathcal{S}_{16}$	16!	16p!	a_{15}, a_{16}	15.6		
17.1	$\mathbb{Z}_{17}$	17	[1^{17}]	$a_{17} : x+1$	1	2	
.2	D_{17}	34	[1,2^8]	$x+1, -x$	$\mathbb{Z}_2$	3, 6	
.3	$F_{17,4}$	68	[1,4^4]	$x+1, 13x$	$\mathbb{Z}_4$	4, 7	
.4	$F_{17,8}$	136	[1,8^2]	$x+1, 9x$	$\mathbb{Z}_8$	5, 8	

Degree $n.i$	Group G	Order $\|G\|$	Trans/Rank	Generators	G_α	min H, $G < H \leq \mathcal{S}_n$	Combinatorial Structures
17.5	$\mathrm{AGL}_1(17)$	272	2t!	$x+1, 3x$	$\mathbb{Z}_{16}$	10	$S_4(4,5,17)$
.6	$L_2(16)$	4080	3t!	a_{17}, b_8^4	16.3	7	$S_6(4,7,17)$
.7	$\mathrm{PZL}_2(16)$	8160	3t	a_{17}, b_8^2	16.6	8	
.8	$\mathrm{P\Gamma L}_2(16)$	16320	3t	a_{17}, b_8	16.10	9	$S_1(3,5,17)$
.9	$\mathcal{A}_{17}$	17!/2	15p!	a_8, a_{17}	16.21	10	
.10	$\mathcal{S}_{17}$	17!	17p!	a_{16}, a_{17}	16.22		
18.1	$L_2(17)$	2448	2p	b_{17}, a_{18}^2	17.4	2, 3	
.2	$\mathrm{PGL}_2(17)$	4896	3t!	a_{18}, b_{17}	17.5	4	
.3	$\mathcal{A}_{18}$	18!/2	16p!	a_9, a_{17}	17.9	4	
.4	$\mathcal{S}_{18}$	18!	18p!	a_{17}, a_{18}	17.10		
19.1	$\mathbb{Z}_{19}$	19	$[1^{19}]$	$a_{19} : x+1$	1	2, 3	$S_1(2,3,19)$
.2	D_{19}	38	$[1,2^9]$	$x+1, -x$	$\mathbb{Z}_2$	4	
.3	$F_{19,3}$	57	$[1,3^6]$	$x+1, 7x$	$\mathbb{Z}_3$	4, 5	$S_1(2,3,19)$
.4	$F_{19,6}$	114	$[1,6^3]$	$x+1, 8x$	$\mathbb{Z}_6$	6	
.5	$F_{19,9}$	171	2*!	$x+1, 4x$	$\mathbb{Z}_9$	6, 7	$S_{112}(5,8,19)$
.6	$\mathrm{AGL}_1(19)$	342	2t!	$x+1, 2x$	$\mathbb{Z}_{18}$	8	
.7	$\mathcal{A}_{19}$	19!/2	17p!	a_9, a_{19}	18.3	8	
.8	$\mathcal{S}_{19}$	19!	19p!	a_{18}, a_{19}	18.4		
20.1	$L_2(19)$	3420	3*	a_{19}, b_{20}^2	19.5	2, 3	$S_{112}(6,9,20)$
.2	$\mathrm{PGL}_2(19)$	6840	3t!	a_{19}, b_{20}	19.6	4	
.3	$\mathcal{A}_{20}$	20!/2	18p!	a_{10}, a_{19}	19.7	4	
.4	$\mathcal{S}_{20}$	20!	20p!	a_{19}, a_{20}	19.8		
21.1	$\mathrm{PGL}_2(7)$	336	$[1,4,8^2]$	a_7, b_2	D_8	9	
.2	$\mathcal{A}_7$	2520	$[1,10^2]$	a_7, c_4^2	10.2	3, 8	
.3	$\mathcal{S}_7$	5040	$[1,10^2]$	a_7, c_4	$\mathcal{S}_5 \times \mathcal{S}_2$	9	
.4	$M_{21} \cong L_3(4)$	20160	2t	a_{21}^3, b_4^2	≈ 16.11	5, 6	
.5	$\mathrm{P\Sigma L}_3(4)$	40320	2t	a_{21}^3, b_4	≈ 16.14	7	
.6	$\mathrm{PGL}_3(4)$	60480	2t	a_{21}, b_4^2	≈ 16.15	7, 8	
.7	$\mathrm{P\Gamma L}_3(4)$	120960	2t	a_{21}, b_4	≈ 16.16	9	$\mathrm{PG}_2(4)$
.8	$\mathcal{A}_{21}$	21!/2	19p!	a_{10}, a_{21}	20.3	9	
.9	$\mathcal{S}_{21}$	21!	21p!	a_{20}, a_{21}	20.4		
22.1	M_{22}	443520	3t	a_{11}, b_{10}^2	21.4	2, 3	$S_1(3,6,22)$
.2	$M_{22} \cdot \mathbb{Z}_2$	887040	3t	a_{11}, b_{10}	21.5	4	
.3	$\mathcal{A}_{22}$	22!/2	20p!	a_{11}, a_{21}	21.8	4	

Degree $n.i$	Group G	Order $\lvert G\rvert$	Trans/Rank	Generators	G_α	min H, $G < H \le \mathcal{S}_n$	Combinatorial Structures
22.4	$\mathcal{S}_{22}$	22!	22p!	a_{21}, a_{22}	21.9		
23.1	$\mathbb{Z}_{23}$	23	$[1^{23}]$	$a_{23} : x+1$	1	2, 3	
.2	D_{23}	46	$[1,2^{11}]$	$x+1, -x$	$\mathbb{Z}_2$	4	
.3	$F_{23,11}$	253	2*	$x+1, 2x$	$\mathbb{Z}_{11}$	4, 5	$S_1(4,5,23)$
.4	$\mathrm{AGL}_1(23)$	506	2t!	$x+1, 5x$	$\mathbb{Z}_{22}$	7	
.5	M_{23}	10200960	4t	a_{23}, b_2	22.1	6	$S_1(4,7,23)$
.6	$\mathcal{A}_{23}$	23!/2	21p!	a_{11}, a_{23}	22.3	7	
.7	$\mathcal{S}_{23}$	23!	23p!	a_{22}, a_{23}	22.4		
24.1	$L_2(23)$	6072	3*	a_{23}, b_{24}^2	23.3	2, 3	$S_1(5,6,24)$
.2	$\mathrm{PGL}_2(23)$	12144	3t!	a_{23}, b_{24}	23.4	5	
.3	M_{24}	244823040	5t	a_{23}, b_2	23.5	4	$S_1(5,8,24)$
.4	$\mathcal{A}_{24}$	24!/2	22p!	a_{12}, a_{23}	23.6	5	
.5	$\mathcal{S}_{24}$	24!	24p!	a_{23}, a_{24}	23.7		
25.1	$F_{25,3}$	75	$[1,3^8]$	a_5, b_{24}^8	$\mathbb{Z}_3$	2, 3	
.2	$F_{25,6}$	150	$[1,6^4]$	a_5, b_{24}^4	$\mathbb{Z}_6$	7,8,9,13	
.3	$V_{25} \cdot \mathcal{S}_3$	150	$[1,3^4,6^2]$	b_{10}, b_{24}^8	$\mathcal{S}_3$	9	
.4	$D_5 \wr \mathcal{S}_2$	200	$[1,4^4,8]$	b_2, b_{20}^2	D_4	11, 22	
.5	$F_{25,8}$	200	$[1,8^3]$	b_{24}^6, c_8	$\mathbb{Z}_8$	10, $14^{c_{10}}$, 15	
.6	$V_{25} \cdot Q_8$	200	$[1,8^3]$	b_4, d_4	Q_8	11, 13	
.7	$F_{25,12}$	300	$[1,12^2]$	a_5, b_{24}^2	$\mathbb{Z}_{12}$	12, 14, 15, 17	
.8	$V_{25} \cdot (\mathbb{Z}_3 \cdot \mathbb{Z}_4)$	300	$[1,12^2]$	b_{24}^4, c_4	$\mathbb{Z}_3 \cdot \mathbb{Z}_4$	12, 20	
.9	$V_{25} \cdot D_6$	300	$[1,6^4]$	b_{24}^4, b_{10}	D_6	12	
.10	$V_{25} \cdot (\mathbb{Z}_8 \cdot_5 \mathbb{Z}_2)$	400	[1,8,16]	b_{20}^2, c_8	$\mathbb{Z}_8 \cdot_5 \mathbb{Z}_2$	16, $18^{c_{20}}$, 25	
.11	$V_{25} \cdot (Q_8 \cdot \mathbb{Z}_2)$	400	$[1,8^3]$	b_{24}^6, b_4, b_2	$Q_8 \cdot \mathbb{Z}_2$	16, 17, 24	
.12	$V_{25} \cdot (\mathcal{S}_3 \times \mathbb{Z}_4)$	600	$[1,12^2]$	b_{24}^2, b_{10}	$\mathcal{S}_3 \times \mathbb{Z}_4$	18, 21	
.13	$V_{25} \cdot (Q_8 \cdot \mathbb{Z}_3)$	600	2t!	b_{24}^4, b_4	$Q_8 \cdot \mathbb{Z}_3$	17, 20	
.14	$\mathrm{AGL}_1(25)$	600	2t!	b_{24}, a_5	$\mathbb{Z}_{24}$	18	
.15	$V_{25} \cdot (\mathbb{Z}_3 \cdot \mathbb{Z}_8)$	600	2t!	b_{24}^2, c_8	$\mathbb{Z}_3 \cdot \mathbb{Z}_8$	18, 19	
.16	$\mathrm{AGL}_1(5) \wr \mathcal{S}_2$	800	[1,8,16]	b_{20}, b_8	$\mathbb{Z}_4 \wr \mathbb{Z}_2$	19, 26	
.17	$V_{25} \cdot G_\alpha$	1200	2t	b_{24}^2, b_{20}^2	$(Q_8 \cdot \mathbb{Z}_2) \cdot \mathbb{Z}_3$	19, 21	
.18	$V_{25} \cdot (\mathbb{Z}_{24} \cdot_5 \mathbb{Z}_2)$	1200	2t	b_{24}, b_{10}	$\mathbb{Z}_{24} \cdot_5 \mathbb{Z}_2$	23	
.19	$V_{25} \cdot G_\alpha$	2400	2t	b_{24}^2, b_{20}	$(Q_8 \cdot \mathbb{Z}_3) \cdot \mathbb{Z}_4$	23	
.20	$\mathrm{ASL}_2(5)$	3000	2t	b_{24}^4, c_{20}^2	$\approx \mathbb{Z}_2 \backslash 6.1$	21	
.21	$\mathcal{A}_{25} \cap \mathrm{AGL}_2(5)$	6000	2t	b_{24}^2, c_{20}	$\approx \mathbb{Z}_4 \backslash 6.1$	23, 27	

Degree $n.i$	Group G	Order $\lvert G\rvert$	Trans/Rank	Generators	G_α	min H, $G < H \leq \mathcal{S}_n$	Combinatorial Structures
25.22	$\mathcal{A}_5 \wr \mathcal{S}_2$	7200	[1,8,16]	b_6, b_{12}^4	16.4	24	
.23	$\mathrm{AGL}_2(5)$	12000	2t	b_{24}, b_{20}	$\mathrm{GL}_2(5)$	28	
.24	$\mathcal{A}_5^2 \cdot V_4$	14400	[1,8,16]	b_6, b_{12}^2	16.8	26, 27	
.25	$\mathcal{A}_5^2 \cdot \mathbb{Z}_4$	14400	[1,8,16]	b_6^2, b_{12}	16.7	26	
.26	$\mathcal{S}_5 \wr \mathcal{S}_2$	28800	[1,8,16]	b_6, b_{12}	16.12	28	
.27	$\mathcal{A}_{25}$	25!/2	23p!	a_{12}, a_{25}	24.4	28	
.28	$\mathcal{S}_{25}$	25!	25p!	a_{24}, a_{25}	24.5		
26.1	$L_2(25)$	7800	2p	a_{26}^2, b_{10}^2	25.7	2,3,4	
.2	$\mathrm{P\Sigma L}_2(25)$	15600	2p	a_{26}^2, b_{10}	25.12	5,6	
.3	$\mathrm{PGL}_2(25)$	15600	3t!	a_{26}, b_{10}^2	25.14	5	
.4	$\mathrm{P\Delta L}_2(25)$	15600	3t!	a_{26}^2, b_{12}	25.15	5	
.5	$\mathrm{P\Gamma L}_2(25)$	31200	3t	a_{26}, b_{10}	25.18	7	
.6	$\mathcal{A}_{26}$	26!/2	24p!	a_{13}, a_{25}	25.27	7	
.7	$\mathcal{S}_{26}$	26!	26p!	a_{25}, a_{26}	25.28		
27.1	$V_{27} \cdot \mathcal{A}_4$	324	[1,4^2,6,12]	a_9, b_6^2	$\mathcal{A}_4$	3,4,5	
.2	$F_{27,13}$	351	[1,13^2]	a_9^3, b_{26}^2	$\mathbb{Z}_{13}$	6,7	$S_1(4,6,27)$
.3	$V_{27} \cdot \mathcal{S}_4$	648	[1,6,8,12]	a_9, b_4	$\mathcal{S}_4$	8,12	
.4	$V_{27} \cdot \mathcal{S}_4$	648	[1,4^2,6,12]	a_9, c_4	$\mathcal{S}_4$	8	
.5	$V_{27} \cdot (V8 \cdot \mathbb{Z}_3)$	648	[1,6,8,12]	a_9, b_6	$V_8 \cdot \mathbb{Z}_3$	8	
.6	$\mathrm{AGL}_1(27)$	702	2t!	a_9^3, b_{26}	$\mathbb{Z}_{26}$	9	
.7	$V_{27} \cdot F_{13,3}$	1053	[1,13^2]	a_9, d_6^2	13.3	9,12	
.8	$V_{27} \cdot (\mathcal{S}_4 \times \mathbb{Z}_2)$	1296	[1,6,8,12]	b_4, c_6	$\mathcal{S}_4 \times \mathbb{Z}_2$	13	
.9	$\mathrm{A\Gamma L}_1(27)$	2106	2t	a_9, d_6	$\mathbb{Z}_{26} \cdot \mathbb{Z}_3$	13	
.10	$U_4(2) \cong S_4(3)$	25920	[1,10,16]	a_9, b_{12}^2	16.9	11	
.11	$U_4(2) \cdot \mathbb{Z}_2$	51840	[1,10,16]	a_9, b_{12}	16.13	14	$\mathrm{srg}(27,10,1,5)$
.12	$\mathrm{ASL}_3(3)$	151632	2t	a_9, b_{24}^2	≈ 13.7	13,14	
.13	$\mathrm{AGL}(3,3)$	303264	2t	a_9, b_{24}	$\approx \mathbb{Z}_2 \backslash 13.7$	15	
.14	$\mathcal{A}_{27}$	27!/2	25p!	a_{13}, a_{27}	26.6	15	
.15	$\mathcal{S}_{27}$	27!	27p!	a_{26}, a_{27}	26.7		
28.1	$\mathrm{PGL}_2(7)$	336	[1,3,6^2,12]	a_7, b_6	D_6	6, 10	
.2	$L_2(8)$	504	[1,9^3]	a_7, b_9^3	D_9	3	
.3	$\mathrm{P\Gamma L}_2(8)$	1512	2t	a_7, b_9	$D_9 \cdot \mathbb{Z}_3$	12	
.4	$U_3(3)$	6048	2t	a_7, b_{12}^2	$T_{27} \cdot \mathbb{Z}_8$	6	

Degree $n.i$	Group G	Order $\|G\|$	Trans/Rank	Generators	G_α	min H, $G < H \le \mathcal{S}_n$	Combinatorial Structures
28.5	$L_2(27)$	9828	2p	a_7, b_{26}^2	27.2	7,9	$S_1(5,7,28)$
.6	$\mathrm{P\Sigma U}_3(3)$	12096	2t	a_7, b_{12}	$T_{27} \cdot (\mathbb{Z}_8 \cdot_3 \mathbb{Z}_2)$	12	
.7	$\mathrm{PGL}_2(27)$	19656	3t!	a_7, b_{26}	27.6	11	
.8	$\mathcal{A}_8$	20160	[1,12,15]	a_7, c_6	$\mathcal{S}_6$	10	
.9	$\mathrm{P\Sigma L}_2(27)$	29484	2p	a_7, c_{12}^2	27.7	11,13	
.10	$\mathcal{S}_8$	40320	[1,12,15]	a_7, b_8	$\mathcal{S}_6 \times \mathbb{Z}_2$	12	
.11	$\mathrm{P\Gamma L}_2(27)$	58968	3t	a_7, c_{12}	27.9	14	
.12	$S_6(2)$	1451520	2p	a_7, d_{12}	27.11	13	
.13	$\mathcal{A}_{28}$	28!/2	26p!	a_{14}, a_{27}	27.14	14	
.14	$\mathcal{S}_{28}$	28!	28p!	a_{27}, a_{28}	27.15		
29.1	$\mathbb{Z}_{29}$	29	$[1^{29}]$	$a_{29} : x+1$	1	2,4	
.2	$F_{29,2}$	58	$[1,2^{14}]$	$x+1, 28x$	$\mathbb{Z}_2$	3,5	
.3	$F_{29,4}$	116	$[1,4^{7}]$	$x+1, 12x$	$\mathbb{Z}_4$	6	
.4	$F_{29,7}$	203	$[1,7^{4}]$	$x+1, 16x$	$\mathbb{Z}_7$	5	
.5	$F_{29,14}$	406	$[1,14^{2}]$	$x+1, 4x$	$\mathbb{Z}_{14}$	6,7	
.6	$\mathrm{AGL}_1(29)$	812	2t!	$x+1, 2x$	$\mathbb{Z}_{28}$	8	
.7	$\mathcal{A}_{29}$	29!/2	27p!	a_{14}, a_{29}	28.13	8	
.8	$\mathcal{S}_{29}$	29!	29p!	a_{28}, a_{29}	28.14		
30.1	$L_2(29)$	12180	2p	a_{29}, b_{30}^2	29.5	2,3	
.2	$\mathrm{PGL}_2(29)$	24360	3t!	a_{29}, b_{30}	29.6	4	
.3	$\mathcal{A}_{30}$	30!/2	28p!	a_{15}, a_{29}	29.7	4	
.4	$\mathcal{S}_{30}$	30!	30p!	a_{29}, a_{30}	29.8		
31.1	$\mathbb{Z}_{31}$	31	$[1^{31}]$	$a_{31} : x+1$	1	2, 3, 4	
.2	$F_{31,2}$	62	$[1,2^{15}]$	$x+1,\ -x$	$\mathbb{Z}_2$	5, 6	
.3	$F_{31,3}$	93	$[1,3^{10}]$	$x+1,\ 5x$	$\mathbb{Z}_3$	5, 7, 9	
.4	$F_{31,5}$	155	$[1,5^{6}]$	$x+1,\ 2x$	$\mathbb{Z}_5$	6, 7, 10	$S_{36}(4,6,31),\ S_{10}(5,6,31)$
.5	$F_{31,6}$	186	$[1,6^{5}]$	$x+1,\ 6x$	$\mathbb{Z}_6$	8	
.6	$F_{31,10}$	310	$[1,10^{3}]$	$x+1,\ 15x$	$\mathbb{Z}_{10}$	8	
.7	$F_{31,15}$	465	$[1,15^{2}]$	$x+1,\ 7x$	$\mathbb{Z}_{15}$	8, 11	
.8	$\mathrm{AGL}_1(31)$	930	2t!	$x+1,\ 3x$	$\mathbb{Z}_{30}$	12	
.9	$L_3(5)$	372000	2t	$a_{31},\ b_{31}$	$\mathrm{AGL}_2(5)$	11	
.10	$L_5(2)$	9999360	2t	$a_{31},\ c_{31}$	$\mathrm{AGL}_4(2)$	11	
.11	$\mathcal{A}_{31}$	31!/2	29p!	$a_{15},\ a_{31}$	30.3	12	

Degree $n.i$	Group G	Order $\|G\|$	Trans/Rank	Generators	G_α	min H, $G < H \leq \mathcal{S}_n$	Combinatorial Structures
31.12	$\mathcal{S}_{31}$	31!	31p!	$a_{30},\ a_{31}$	30.4		
32.1	$\mathrm{AGL}_1(32)$	992	2p!	a_{31}, b_{10}^5	31.1	2	
.2	$\mathrm{A\Gamma L}_1(32)$	4960	3*!	a_{31}, b_{10}	31.4	5	$S_{36}(5,7,32)$,$S_{10}(6,7,32)$
.3	$L_2(31)$	14880	3*	a_{31}, b_{32}^2	31.7	4, 6	
.4	$\mathrm{PGL}_2(31)$	29760	3t!	a_{31}, b_{32}	31.8	7	
.5	$\mathrm{AGL}_5(2)$	319979520	3t	a_{31}, b_{31}	31.10	6	
.6	$\mathcal{A}_{32}$	32!/2	30p!	a_{16}, a_{31}	31.11	7	
.7	$\mathcal{S}_{32}$	32!	32p!	a_{31}, a_{32}	31.12		
33.1	$L_2(32)$	32736	3p!	a_{33}, b_{15}^5	32.1	2	
.2	$\mathrm{P\Gamma L}_2(32)$	163680	4*	a_{33}, b_{15}	32.2	3	$S_{36}(6,8,33)$,$S_{10}(7,8,33)$
.3	$\mathcal{A}_{33}$	33!/2	31p!	a_{16}, a_{33}	32.6	4	
.4	$\mathcal{S}_{33}$	33!	33p!	a_{32}, a_{33}	32.7		
34.1	$\mathcal{A}_{34}$	34!/2	32p!	a_{17}, a_{33}	33.3	2	
.2	$\mathcal{S}_{34}$	34!	34p!	a_{33}, a_{34}	33.4		
35.1	$\mathcal{A}_7$	2520	[1,4,12,18]	$a_{15,5}^3, b_{12}^2$	$\approx \mathbb{Z}_2\backslash 9.1$	2, 3	
.2	$\mathcal{S}_7$	5040	[1,4,12,18]	$a_{15,5}^3, b_{12}$	≈ 9.5	4	
.3	$\mathcal{A}_8$	20160	[1,16,18]	$a_{15,5}, b_{12}^2$	16.8	4	
.4	$\mathcal{S}_8$	40320	[1,16,18]	$a_{15,5}, b_{12}$	16.12	5	srg$(35,16,6,8)$
.5	$\mathcal{A}_{35}$	35!/2	33p!	a_{17}, a_{35}	34.1	6	
.6	$\mathcal{S}_{35}$	35!	35p!	a_{34}, a_{35}	34.2		
36.1	$L_2(8)$	504	[1,7^3,14]	a_{36}^{12}, b_8^4	D_7	5	
.2	$\mathrm{PGL}_2(9)$	720	[1,5,10^3]	b_{10}, c_8^2	D_{10}	4,15	
.3	M_{10}	720	[1,5,10,20]	b_{10}^2, c_8	$F_{5,4}$	4,$15^{b_{12}}$	
.4	$\mathrm{P\Gamma L}_2(9)$	1440	[1,5,10,20]	b_{10}, c_8	$F_{5,4} \times \mathbb{Z}_2$	17	
.5	$\mathrm{P\Gamma L}_2(8)$	1512	[1,14,21]	a_{36}^4, b_8^4	≈ 7.4	14,20	
.6	$U_3(3)$	6048	[1,7^2,21]	a_{36}^{12}, c_{12}^2	7.5	8	
.7	$\mathcal{A}_5 \wr \mathcal{S}_2$	7200	[1,10,25]	b_{10}, b_{12}^4	25.4	9,15	
.8	$\mathrm{P\Sigma U}_3(3)$	12096	[1,14,21]	a_{36}^{12}, c_{12}	21.1	20	srg$(36,14,4,6)$
.9	$\mathcal{A}_5^2 \cdot V_4$	14400	[1,10,25]	b_{10}, b_{12}^2	25.11	12,17	
.10	$\mathcal{A}_5^2 \cdot \mathbb{Z}_4$	14400	[1,10,25]	b_{10}^2, b_{12}	25.10	12,18	
.11	$S_4(3) \cong U_4(2)$	25920	[1,15,20]	a_{36}^4, b_6^2	15.2	13	
.12	$\mathcal{S}_5 \wr \mathcal{S}_2$	28800	[1,10,25]	b_{10}, b_{12}	25.16	19	
.13	$U_4(2) \cdot \mathbb{Z}_2$	51840	[1,15,20]	a_{36}^4, b_6	$\mathcal{S}_6 \times \mathcal{S}_2$	20	srg$(36,15,6,6)$

Degree $n.i$	Group G	Order $\|G\|$	Trans/Rank	Generators	G_α	min H, $G < H \leq \mathcal{S}_n$	Combinatorial Structures
36.14	$\mathcal{A}_9$	181440	[1,14,21]	a_{36}^4, b_8^2	21.3	16, 21	
.15	$\mathcal{A}_6 \wr \mathcal{S}_2$	259200	[1,10,25]	a_{36}^{12}, b_{10}	25.22	17	
.16	$\mathcal{S}_9$	362880	[1,14,21]	a_{36}^4, b_8	$\mathcal{S}_7 \times \mathcal{S}_2$	22	
.17	$\mathcal{A}_6^2 \cdot V_4$	518400	[1,10,25]	a_{36}^6, b_{10}	25.24	19	
.18	$\mathcal{A}_6^2 \cdot \mathbb{Z}_4$	518400	[1,10,25]	a_{36}^3, b_{10}^2	25.25	19	
.19	$\mathcal{S}_6 \wr \mathcal{S}_2$	1036800	[1,10,25]	a_{36}^3, b_{10}	25.26	22	
.20	$S_6(2)$	1451520	2p	a_{36}^4, c_{12}	35.4	21	
.21	$\mathcal{A}_{36}$	36!/2	34p!	a_{18}, a_{35}	35.5	22	
.22	$\mathcal{S}_{36}$	36!	36p!	a_{35}, a_{36}	35.6		
37.1	$\mathbb{Z}_{37}$	37	$[1^{37}]$	$a_{37} : x+1$	1	2,3	
.2	$F_{37,2}$	74	$[1, 2^{18}]$	$x+1, -x$	$\mathbb{Z}_2$	4,5	
.3	$F_{37,3}$	111	$[1, 3^{12}]$	$x+1, 26x$	$\mathbb{Z}_3$	5,6	
.4	$F_{37,4}$	148	$[1, 4^9]$	$x+1, 31x$	$\mathbb{Z}_4$	7	
.5	$F_{37,6}$	222	$[1, 6^6]$	$x+1, 27x$	$\mathbb{Z}_6$	7,8	
.6	$F_{37,9}$	333	$[1, 9^4]$	$x+1, 16x$	$\mathbb{Z}_9$	8,11	
.7	$F_{37,12}$	444	$[1, 12^3]$	$x+1, 8x$	$\mathbb{Z}_{12}$	9	
.8	$F_{37,18}$	666	$[1, 18^2]$	$x+1, 4x$	$\mathbb{Z}_{18}$	9,10	
.9	$\mathrm{AGL}_1(37)$	1332	2t!	$x+1, 2x$	$\mathbb{Z}_{36}$	11	
.10	$\mathcal{A}_{37}$	37!/2	35p!	a_{18}, a_{37}	36.21	11	
.11	$\mathcal{S}_{37}$	37!	37p!	a_{36}, a_{37}	36.22		
38.1	$L_2(37)$	25308	2p	a_{37}, b_{38}^2	37.8	2,3	
.2	$\mathrm{PGL}_2(37)$	50616	3t!	a_{37}, b_{38}	37.9	4	
.3	$\mathcal{A}_{38}$	38!/2	36p!	a_{19}, a_{37}	37.10	4	
.4	$\mathcal{S}_{38}$	38!	38p!	a_{36}, a_{37}	37.11		
39.1	$\mathcal{A}_{39}$	39!/2	37p!	a_{19}, a_{39}	38.3	2	
.2	$\mathcal{S}_{39}$	39!	39p!	a_{38}, a_{39}	38.4		
40.1	$S_4(3) \cong U_4(2)$	25920	[1,12,27]	a_{40}^8, b_5	$T_{27} \cdot (Q_8 \cdot \mathbb{Z}_3)$	3	
.2	$S_4(3) \cong U_4(2)$	25920	[1,12,27]	a_{40}^8, b_{10}^2	27.4	4	
.3	$U_4(2) \cdot \mathbb{Z}_2$	51840	[1,12,27]	a_{40}^4, b_5	$T_{27} \cdot (Q_8 \cdot \mathcal{S}_3)$	5	
.4	$U_4(2) \cdot \mathbb{Z}_2$	51840	[1,12,27]	a_{40}^8, b_{10}	27.8	8	
.5	$L_4(3)$	6065280	2t	a_{40}^2, b_5	$\mathrm{ASL}_3(3)$	6,7	
.6	$\mathrm{PGL}_4(3)$	12130560	2t	a_{40}, b_5	$\mathrm{AGL}_3(3)$	8	
.7	$\mathcal{A}_{40}$	40!/2	38p!	a_{20}, a_{39}	39.1	8	

Degree $n.i$	Group G	Order $\|G\|$	Trans/Rank	Generators	G_α	min H, $G < H \le \mathcal{S}_n$	Combinatorial Structures
40.8	$\mathcal{S}_{40}$	40!	40p!	a_{39}, a_{40}	39.2		
41.1	$\mathbb{Z}_{41}$	41	$[1^{41}]$	$a_{41} : x+1$	1	2,4	
.2	$F_{41,2}$	82	$[1,2^{20}]$	$x+1, 40x$	$\mathbb{Z}_2$	3,6	
.3	$F_{41,4}$	164	$[1,4^{10}]$	$x+1, 32x$	$\mathbb{Z}_4$	5,7	
.4	$F_{41,5}$	205	$[1,5^{8}]$	$x+1, 10x$	$\mathbb{Z}_5$	6	
.5	$F_{41,8}$	328	$[1,8^{5}]$	$x+1, 27x$	$\mathbb{Z}_8$	8	
.6	$F_{41,10}$	410	$[1,10^{4}]$	$x+1, 25x$	$\mathbb{Z}_{10}$	7	
.7	$F_{41,20}$	820	$[1,20^{2}]$	$x+1, 36x$	$\mathbb{Z}_{20}$	8,9	
.8	$\mathrm{AGL}_1(41)$	1640	2t!	$x+1, 6x$	$\mathbb{Z}_{40}$	10	
.9	$\mathcal{A}_{41}$	41!/2	39p!	a_{20}, a_{41}	40.7	10	
.10	$\mathcal{S}_{41}$	41!	41p!	a_{40}, a_{41}	40.8		
42.1	$L_2(41)$	34440	2p	a_{41}, b_{42}^2	41.7	2,3	
.2	$\mathrm{PGL}_2(41)$	68880	3t!	a_{41}, b_{42}	41.8	4	
.3	$\mathcal{A}_{42}$	42!/2	40p!	a_{21}, a_{41}	41.9	4	
.4	$\mathcal{S}_{42}$	42!	42p!	a_{41}, a_{42}	41.10		
43.1	$\mathbb{Z}_{43}$	43	$[1^{43}]$	$a_{43} : x+1$	1	2,3,5	
.2	$F_{43,2}$	86	$[1,2^{21}]$	$x+1, 42x$	$\mathbb{Z}_2$	4,6	
.3	$F_{43,3}$	129	$[1,3^{14}]$	$x+1, 36x$	$\mathbb{Z}_3$	4,7	
.4	$F_{43,6}$	258	$[1,6^{7}]$	$x+1, 37x$	$\mathbb{Z}_6$	8	
.5	$F_{43,7}$	301	$[1,7^{6}]$	$x+1, 41x$	$\mathbb{Z}_7$	6,7	
.6	$F_{43,14}$	602	$[1,14^{3}]$	$x+1, 27x$	$\mathbb{Z}_{14}$	8	
.7	$F_{43,21}$	903	$[1,21^{2}]$	$x+1, 9x$	$\mathbb{Z}_{21}$	8,9	
.8	$\mathrm{AGL}_1(43)$	1806	2t!	$x+1, 3x$	$\mathbb{Z}_{42}$	10	
.9	$\mathcal{A}_{43}$	43!/2	41p!	a_{21}, a_{43}	42.3	10	
.10	$\mathcal{S}_{43}$	43!	43p!	a_{42}, a_{43}	42.4		
44.1	$L_2(43)$	39732	2p	a_{43}, b_{44}^2	43.7	2,3	
.2	$\mathrm{PGL}_2(43)$	79464	3t!	a_{43}, b_{44}	43.8	4	
.3	$\mathcal{A}_{44}$	44!/2	42p!	a_{22}, a_{43}	43.9	4	
.4	$\mathcal{S}_{44}$	44!	44p!	a_{43}, a_{44}	43.10		
45.1	$\mathrm{PGL}_2(9)$	720	$[1,4,8^3,16]$	b_8^2, b_{10}	D_8	3	
.2	M_{10}	720	$[1,4,8,16^2]$	b_8, b_{10}^2	$\mathbb{Z}_8 \cdot_3 \mathbb{Z}_2$	3, 6	
.3	$\mathrm{P\Gamma L}_2(9)$	1440	$[1,4,8,16^2]$	b_8, b_{10}	$\mathbb{Z}_8 \cdot Aut\ \mathbb{Z}_8$	7	
.4	$S_4(3) \cong U_4(2)$	25920	[1,12,32]	b_{10}^2, d_8^2	$\mathbb{Z}_2 \backslash 16.4$	5, 8	

Degree $n.i$	Group G	Order $\|G\|$	Trans/Rank	Generators	G_α	min H, $G < H \le \mathcal{S}_n$	Combinatorial Structures
45.5	$U_4(2) \cdot \mathbb{Z}_2$	51840	[1,12,32]	b_{10}^2, d_8	$\mathbb{Z}_2 \backslash 16.8$	9	$\mathrm{srg}(45, 12, 3, 3)$
.6	$\mathcal{A}_{10}$	1814400	[1,16,28]	b_{10}^2, c_8^2	28.10	7	
.7	$\mathcal{S}_{10}$	3628800	[1,16,28]	b_{10}, c_8	$\mathcal{S}_8 \times \mathcal{S}_2$	8	
.8	$\mathcal{A}_{45}$	45!/2	43p!	a_{22}, a_{45}	44.3	9	
.9	$\mathcal{S}_{45}$	45!	45p!	a_{44}, a_{45}	44.4		
46.1	$\mathcal{A}_{46}$	46!/2	44p!	a_{23}, a_{45}	45.8	2	
.2	$\mathcal{S}_{46}$	46!	46p!	a_{45}, a_{46}	45.9		
47.1	$\mathbb{Z}_{47}$	47	$[1^{47}]$	$a_{47} : x+1$	1	2, 3	
.2	$F_{47,2}$	94	$[1,2^{23}]$	$x+1, 46x$	$\mathbb{Z}_2$	4	
.3	$F_{47,23}$	1081	$[1,23^2]$	$x+1, 25x$	$\mathbb{Z}_{23}$	4, 5	$S_1(4, 5, 47)$
.4	$\mathrm{AGL}_1 47$	2162	2t!	$x+1, 5x$	$\mathbb{Z}_{46}$	6	
.5	$\mathcal{A}_{47}$	47!/2	45p!	a_{23}, a_{47}	46.1	6	
.6	$\mathcal{S}_{47}$	47!	47p!	a_{46}, a_{47}	46.2		
48.1	$L_2(47)$	51888	2p	a_{48}^2, b_{47}	47.3	2,3	$S_1(5, 6, 48)$
.2	$\mathrm{PGL}_2(47)$	103776	3t!	a_{48}, b_{47}	47.4	4	
.3	$\mathcal{A}_{48}$	48!/2	46p!	a_{24}, a_{47}	47.5	4	
.4	$\mathcal{S}_{48}$	48!	48p!	a_{47}, a_{48}	47.6		
49.1	$F_{49,4}$	196	$[1,4^{12}]$	a_7, b_{48}^{12}	$\mathbb{Z}_4$	3,4,5,6,7	
.2	$V_{49} \cdot \mathcal{S}_3$	294	$[1,3^6, 6^5]$	b_{42}^3, b_{14}	$\mathcal{S}_3$	8, 12	
.3	$F_{49,8}$	392	$[1,8^6]$	a_7, b_{48}^6	$\mathbb{Z}_8$	9,10,11,13	
.4	$V_{49} \cdot Q_8$	392	$[1,8^6]$	b_{48}^{12}, b_{12}^3	Q_8	9,14,15,16	
.5	$D_7 \wr \mathcal{S}_2$	392	$[1,4^6, 8^3]$	b_{14}, b_{48}^{12}	D_4	11,17,18	
.6	$V_{49} \cdot (\mathbb{Z}_3 \cdot \mathbb{Z}_4)$	588	$[1,12^4]$	b_{48}^{12}, d_6	$\mathbb{Z}_3 \cdot \mathbb{Z}_4$	$17, 20, 22^{b_{42}}, 30$	
.7	$F_{49,12}$	588	$[1,12^4]$	a_7, b_{48}^4	$\mathbb{Z}_{12}$	13,14,18,20	
.8	$V_{49} \cdot D_6$	588	$[1,6^6, 12]$	b_{42}^3, d_6	D_6	17,21	
.9	$V_{49} \cdot (\mathbb{Z}_2 \backslash D_4)$	784	$[1,16^3]$	b_{48}^6, b_{12}^3	$\mathbb{Z}_2 \backslash D_4$	19,22,23	
.10	$F_{49,16}$	784	$[1,16^3]$	a_7, b_{48}^3	$\mathbb{Z}_{16}$	19,24	
.11	$V_{49} \cdot D_8$	784	$[1,8^6]$	b_{14}, b_{48}^6	D_8	19, 25	
.12	$V_{49} \cdot G_\alpha \cong 7.3 \wr \mathcal{S}_2$	882	$[1,6^2, 9^2,18]$	b_{14}, c_6^2	$\mathbb{Z}_3 \wr \mathcal{S}_2$	21, 33	
.13	$F_{49,24}$	1176	$[1,24^2]$	a_7, b_{48}^2	$\mathbb{Z}_{24}$	23, 24, 25	
.14	$V_{49} \cdot (Q_8 \times \mathbb{Z}_3)$	1176	$[1,24^2]$	b_{48}^4, b_{12}	$Q_8 \times \mathbb{Z}_3$	23, 26	
.15	$V_{49} \cdot (Q_8 \cdot \mathbb{Z}_3)$	1176	$[1,24^2]$	b_{48}^{12}, b_3	$Q_8 \cdot \mathbb{Z}_3$	22,26	
.16	$V_{49} \cdot (Q_8 \cdot \mathbb{Z}_3)$	1176	$[1,8^3,24]$	b_{48}^{12}, b_6	$Q_8 \cdot \mathbb{Z}_3$	26	

Degree $n.i$	Group G	Order $\|G\|$	Trans/Rank	Generators	G_α	min H, $G < H \le \mathcal{S}_n$	Combinatorial Structures
49.17	$V_{49} \cdot (\mathbb{Z}_3 \cdot D_4)$	1176	[1,12^4]	b_{48}^{12}, b_{42}^3	$\mathbb{Z}_3 \cdot D_4$	27,31	
.18	$V_{49} \cdot (D_4 \times \mathbb{Z}_3)$	1176	[1,12^2,24]	b_{14}, b_{48}^4	$D_4 \times \mathbb{Z}_3$	25,27	
.19	$V_{49} \cdot (\mathbb{Z}_{16} \cdot_7 \mathbb{Z}_2)$	1568	[1,16^3]	b_{14}, b_{48}^3	$\mathbb{Z}_{16} \cdot_7 \mathbb{Z}_2$	28,31	
.20	$V_{49} \cdot G_\alpha$	1764	[1,12,36]	b_{48}^4, c_6	$\mathbb{Z}_3 \times (\mathbb{Z}_3 \cdot \mathbb{Z}_4)$	27,29$^{b_{42}}$, 32, 37	
.21	$V_{49} \cdot (\mathcal{S}_3 \times \mathbb{Z}_6)$	1764	[1,12,18^2]	b_{14}, c_6	$\mathcal{S}_3 \times \mathbb{Z}_6$	27,36	
.22	$V_{49} \cdot (\mathbb{Z}_2 \backslash \mathcal{S}_4)$	2352	2t!	b_{48}^6, b_8^2	$\mathbb{Z}_2 \backslash \mathcal{S}_4$	29,30	
.23	$V_{49} \cdot G_\alpha$	2352	2t!	b_{48}^2, b_{12}	$\mathbb{Z}_3 \times (\mathbb{Z}_2 \backslash D_4)$	28,29	
.24	$\mathrm{AGL}_1(49)$	2352	2t!	a_7, b_{48}	$\mathbb{Z}_{48}$	28	
.25	$\mathrm{A\Sigma L}_1(49)$	2352	[1,24^2]	b_{48}^2, b_{14}	$\mathbb{Z}_{24} \cdot_7 \mathbb{Z}_2$	28	
.26	$V_{49} \cdot G_\alpha$	3528	[1,24^2]	b_{48}^4, b_6	$(Q_8 \cdot \mathbb{Z}_3) \times \mathbb{Z}_3$	29	
.27	$7.4 \wr \mathcal{S}_2$	3528	[1,12,36]	b_{48}^4, b_{42}^3	$\mathbb{Z}_6 \wr \mathcal{S}_2$	34,38	
.28	$\mathrm{A\Gamma L}_1(49)$	4704	2t	b_{48}, b_{14}	$\mathbb{Z}_{48} \cdot_7 \mathbb{Z}_2$	34	
.29	$V_{49} \cdot G_\alpha$	7056	2t	b_{48}^2, b_8^2	$(\mathbb{Z}_2 \backslash \mathcal{S}_4) \times \mathbb{Z}_3$	32	
.30	$\mathrm{ASL}_2(7)$	16464	2t	b_{48}^6, b_8	$\mathrm{SL}_2(7)$	31,32	
.31	$\mathrm{AS}^{\pm}L_2(7)$	32928	2t	b_{48}^3, b_{42}^3	$(\pm 1) \cdot SL_2(7)$	34	
.32	$\mathrm{AS}^2L_2(7)$	49392	2t	b_{48}^2, b_{42}^2	$\langle 2 \rangle \times SL_2(7)$	34,39	
.33	$L_3(2) \wr \mathcal{S}_2 \cong 7.5 \wr \mathcal{S}_2$	56448	[1,12,36]	b_{24}^{12}, b_{42}^3	$\mathcal{S}_4 \wr \mathcal{S}_2$	35	
.34	$\mathrm{AGL}_2(7)$	98784	2t	b_{48}, b_{42}	$\mathrm{GL}_2(7)$	40	
.35	$\mathcal{A}_7 \wr \mathcal{S}_2$	12700800	[1,12,36]	b_{24}^4, b_{42}^3	36.15	36	
.36	$(\mathcal{A}_7 \times \mathcal{A}_7) \cdot V_4$	25401600	[1,12,36]	b_{24}^2, b_{42}^3	36.17	38	
.37	$(\mathcal{A}_7 \times \mathcal{A}_7) \cdot \mathbb{Z}_4$	25401600	[1,12,36]	b_{24}, b_{42}^6	36.18	38,39	
.38	$\mathcal{S}_7 \wr \mathcal{S}_2$	50803200	[1,12,36]	b_{24}, b_{42}^3	36.19	40	
.39	$\mathcal{A}_{49}$	49!/2	47p!	a_{24}, a_{49}	48.3	40	
.40	$\mathcal{S}_{49}$	49!	49p!	a_{48}, a_{49}	48.4		
50.1	$L_2(49)$	58800	2p	a_7, b_{50}^2	49.13	2,3,4	
.2	$\mathrm{P\Delta L}_2(49)$	117600	3t!	a_7, b_{16}	49.23	6,8	
.3	$\mathrm{PGL}_2(49)$	117600	3t!	a_7, b_{50}	49.24	6	
.4	$\mathrm{P\Sigma L}_2(49)$	117600	2p	b_{14}, b_{50}^2	49.25	6	
.5	$\mathrm{PSU}_3(5^2)$	126000	[1,7,42]	a_7, b_8^2	7.6	7	
.6	$\mathrm{P\Gamma L}_2(49)$	235200	3t	b_{14}, b_{50}	49.28	9	
.7	$\mathrm{P\Sigma U}_3(5^2)$	252000	[1,7,42]	a_7, b_8	7.7	8	$\mathrm{srg}(50, 7, 0, 1)$
.8	$\mathcal{A}_{50}$	50!/2	48p!	a_{25}, a_{49}	49.39	9	
.9	$\mathcal{S}_{50}$	50!	50p!	a_{49}, a_{50}	49.40		

8.63 Table Generators for primitive groups of degree at most 50.

Deg	Elt	Permutation	Elt	Permutation	Elt	Permutation	Elt	Permutation
6	b_3	(1 3 4)(2 5 6)	b_6	(1 5 2 4 3 6)				
7	b_7	(1 4 6 2 3 7 5)						
8	b_6	(1 2 8 5 7 6)(3 4)	b_7	(1 5 6 8 2 4 3)	b_8	(1 2 8 3 4 7 6 5)	c_7	(1 8 7 2 5 3 6)
	d_7	(1 8 2 7 5 4 6)						
9	b_3	(1 5 9)(2 3 8)(4 7 6)	b_4	(1 6 4 5)(2 9 8 7)	b_6	(1 9 3 4 7 5)(2 8)	b_8	(1 7 5 2 8 4 3 9)
	b_9	(1 8 5 2 9 4 7 3 6)	c_4	(1 9 8 2)(3 7 5 4)	c_6	(1 2 4 6 3 9)(5 7)	c_8	(2 9 3 6 7 5 4 8)
	d_4	(1 6 4 5)(2 9 8 7)	d_6	(1 5 6 2 8 4)(3 9)	e_4	(1 3 6 2)(4 5 9 7)	e_6	(1 7 5 6 9 4)(2 8 3)
	f_6	(1 4 3)(2 8 6 7 9 5)	g_6	(1 7 8 9 6 3)(2 5 4)				
10	b_3	(1 6 8)(2 7 10)(3 9 5)	b_5	(1 10 4 7 5)(2 8 6 9 3)	b_6	(1 8 6)(2 3 7 9 10 5)	b_8	(1 8 6 2 9 5 3 10)(4 7)
	b_{10}	(1 2 5 8 9 7 4 10 6 3)	c_5	(1 3 4 5 7)(2 10 9 8 6)	c_6	(1 8 9 3 5 6)(2 7 4)	c_8	(1 6 10 9 3 8 4 5)(2 7)
	d_5	(1 10 9 6 4)(2 8 3 5 7)	d_8	(1 7 2 6 5 9 4 10)(3 8)				

Deg	Elt	Permutation	Elt	Permutation
11	b_5	(1 11 10 3 6)(2 5 4 9 7)	b_{11}	(1 2 7 9 4 3 6 11 5 8 10)
12	b_{11}	(1 12 10 3 7 5 8 4 2 11 9)	c_{11}	(2 3 7 8 10 4 5 9 6 11 12)
	d_{11}	(1 2 10 11 9 5 4 12 8 7 3)	b_{12}	(1 7 2 5 12 6 3 11 9 10 8 4)
13	b_{13}	(1 3 4 6 8 10 7 13 2 5 12 11 9)		
14	b_{13}	(1 9 4 3 8 10 13 5 11 14 2 7 6)		
15	b_2	(1 9)(2 5)(4 15)(8 13)(10 11)(12 14)	b_7	(2 6 3 8 12 4 9)(5 7 13 11 10 14 15)
	c_2	(1 5)(6 13)(7 8)(10 12)		
16	b_3	(2 16 4)(3 14 9)(5 6 11)(7 12 8)	b_4	(2 5)(3 9 8 13)(4 6 11 16)(7 14 10 12)
	b_6	(1 8 10 15 7 3)(2 11 4)(6 9 13 16 14 12)	b_7	(1 2 11 12 7 16 13)(3 15 10 9 8 6 5)
	b_{15}	(1 2 10 14 11 8 5 12 3 13 9 6 15 4 7)	c_6	(1 3 13)(2 14 16 7 5 15)(4 11 10 8 9 12)
17	b_8	(1 15 13 10 16 12 17 6)(2 8 7 4 9 11 5 14)	b_{10}	(1 16 8 4 2 14 9 12 5 7)(3 6 15 11 13)(10 17)
	b_{17}	(1 11 6 12 17 4 7 10 14 2 8 3 13 5 16 15 9)		
18	b_{17}	(1 12 13 8 2 15 16 9 7 11 10 6 14 4 18 17 3)		
20	b_{20}	(1 3 5 17 18 10 16 19 13 11 2 20 4 14 12 6 9 15 7 8)		
21	b_2	(2 17)(3 12)(4 19)(5 16)(6 13)(7 18)(9 21)(11 14)(15 20)	b_4	(1 14 8 12)(3 21 7 18)(4 20 16 19)(5 11)(9 17 10 15)
	c_4	(1 21 4 10)(2 9 11 3)(5 12 15 14)(7 13 20 8)(17 18)		
22	b_{10}	(1 15 17 2 21 14 7 3 22 10)(4 12 11 6 16 13 9 19 20 5)(8 18)		
23	b_2	(1 4)(2 6)(8 23)(9 10)(11 16)(12 19)(13 21)(14 20)		
24	b_2	(6 7)(8 19)(9 14)(10 22)(11 12)(13 24)(15 17)(20 23)		
	b_{24}	(1 2 24 3 4 15 19 22 18 16 8 7 17 23 14 5 11 21 20 12 10 6 9 13)		
25	b_2	(1 18)(2 23)(4 8)(5 13)(6 19)(7 24)(10 14)(11 20)(12 25)(17 21)		

Deg	Elt	Permutation
25	b_4	(2 3 5 4)(6 16 21 11)(7 18 25 14)(8 20 24 12)(9 17 23 15)(10 19 22 13)
	b_6	(1 9 25)(2 14 22 11 7 15)(3 19 23 16 8 20)(4 24 21 6 10 5)(13 17)
	b_8	(1 2 7 9 19 18 13 11)(3 12 6 4 17 8 14 16)(5 22 10 24 20 23 15 21)
	b_{10}	(1 3 5 2 4)(6 22 10 21 9 25 8 24 7 23)(11 16 15 20 14 19 13 18 12 17)
	b_{12}	(1 10 17)(2 5 20 16 6 7)(3 15 19 21 8 12 4 25 18 11 9 22)(13 14 24 23)
	b_{20}	(1 9 17 15 3 6 19 12 5 8 16 14 2 10 18 11 4 7 20 13)(21 24 22 25 23)
	b_{24}	(1 4 17 9 20 22 7 8 14 18 15 24 19 16 3 11 5 23 13 12 6 2 10 21)
25	c_4	(2 3 5 4)(6 19 21 13)(7 16 25 11)(8 18 24 14)(9 20 23 12)(10 17 22 15)
	c_8	(2 6 3 11 5 21 4 16)(7 8 13 15 25 24 19 17)(9 18 12 10 23 14 20 22)
	c_{20}	(2 7 21 12 15 3 13 16 23 24 5 25 6 20 17 4 19 11 9 8)(10 14 22 18)
	d_4	(1 7 3 22)(4 17 5 12)(6 8 23 21)(9 18 25 11)(10 13 24 16)(14 19 20 15)
26	b_{10}	(1 17 16 9 21 15 12 18 14 6)(2 13 19 26 24)(3 25 10 5 4 23 11 8 20 22)
	b_{12}	(1 22 14 25 11 23 17 7 18 5 10 8)(2 16 9 3 4 20 15 12 26 13 21 19)(6 24)
27	b_4	(1 18 5 3)(6 17 26 12)(7 11 21 20)(8 19 27 16)(9 15 22 10)(13 25 23 14)
	b_6	(1 5 17 22 15 6)(2 10 14 18 13 9)(3 27 12 11 26 8)(4 21 25 19 23 7)(20 24)
	b_{12}	(1 25 27 5 13 21)(2 15 18 16)(3 23 17 12)(4 19 8 9 20 14 6 26 11 10 7 22)
	b_{24}	(1 27 21 13 4 10 18 19 9 14 17 15 3 12 2 24 26 22 25 8 20 5 16 11)(6 7 23)
	b_{26}	(1 10 18 5 20 2 27 23 25 3 4 24 21 16 7 26 6 15 9 14 12 19 8 13 17 11)
	c_4	(1 2 18 15)(3 22 13 7)(4 19 26 8)(5 11 23 9)(10 14)(12 17 27 16)(20 25 21 24)
27	c_6	(1 11 26 19 9 24)(2 14 21 15 6 20)(3 16 27 17 13 23)(4 18 22 25 8 7)(5 12)
	d_6	(1 27 19 14 9 20)(2 17 21 16 7 6)(3 11 15 24 4 12)(5 8 26 13 10 22)(23 25)
28	b_6	(1 23 24 4 18 19)(2 8 7 20 21 22)(3 25 16 17 5 14)(6 9 28 15 26 13)(10 12 27)
	b_8	(1 13 26 20 16 23 7 17)(2 25 21 5 12 19 4 28)(3 9 18 6 10 24 22 11)(8 14 15 27)
	b_9	(1 27 22 26 10 23 13 12 3)(2 21 7 24 15 14 25 16 19)(4 5 17 9 18 6 28 8 20)
	b_{12}	(1 11 5 15 14 19 28 10 17 24 2 6)(3 18 13 22 16 7 25 23 4 21 20 8)(9 26 27)
	b_{26}	(1 22 4 27 23 24 5 3 7 8 21 9 25 2 26 16 20 18 10 11 14 19 12 15 17 6)
	c_6	(1 3 14)(2 27)(4 26)(5 20 9 11 19 21)(6 12 17)(8 18)(10 23 16 28 15 25)(13 22)
	c_{12}	(1 24 13 8 10 15 16 7 19 14 4 27)(2 18 11 17 23 21 6 3 9 20 22 12)(5 28 25 26)
	d_{12}	(1 14 21 20 23 4 19 22 7 25 27 9)(3 28 5 11)(6 17 26 10 16 24)(12 13 15 18)
30	b_{30}	(1 20 19 6 28 8 3 15 23 12 13 11 16 27 4 10 21 26 24 25 14 22 5 29 9 2 18 17 7 30)
31	b_{31}	(1 13 15 7 30 10 19 23 3 18 29 6 9 14 25 2 27 28 26 4 21 24 16 5 17 22 12 20 31 11 8)
	c_{31}	(1 26 4 24 19 13 10 20 23 6 17 16 15 7 22 11 12 21 9 8 14 18 5 25 3 2 27 31 30 29 28)
32	b_{10}	(1 21 25 13 6 14 10 26 32 8)(2 12 19 29 30 5 31 4 7 3)(9 16 11 27 15 23 24 18 17 20)(22 28)
	b_{31}	(1 5 19 9 23 3 24 15 2 21 12 25 6 20 28 8 11 17 14 22 7 26 29 32 4 30 31 13 10 16 18)

Deg	Elt	Permutation
32	b_{32}	(1 29 23 21 13 25 27 11 12 31 30 6 14 9 32 7 2 10 17 16 4 5 20 22 3 26 24 18 15 19 8 28)
33	b_{15}	(1 20 29 24 7 32 17 16 8 18 30 6 10 13 23)(2 22 14 33 4 21 31 11 9 28 15 19 25 12 5)(3 26 27)
35	b_{12}	(1 21 31 28 9 12 29 15 3 25 33 27)(2 5 30 10 26 34 16 22 35 17 7 18)(4 14 6 8 13 32)(11 20 19 23)
	$a_{15,5}$	(a_{15} for degree 30) $\cdot$ (31 32 33 34 35)
36	b_6	(1 10 16 20 28 29)(2 15 35 13 12 3)(4 19 27)(6 26 34)(7 31 25)(8 14)(9 17 24 21 22 23)(11 32 30)
	b_8	(1 17 33 30 2 28 25 24)(3 22 31 10 4 6 36 27)(5 35 11 34 7 23 20 13)(8 29 15 9 18 26 32 19)
	b_{10}	(1 11 33 28 22 15 23 14 20 13)(2 25 31 30 9)(3 8 6 26 18 19 36 16 21 32)(4 27 17 34 35 12 10 29 7 5)
	b_{12}	(1 36 6 21 14 22 26 2 34 31 8 28)(3 35 32 10 11 24 23 4 19 17 18 7)(5 30 16 9 29 12 13 33 25 15 20 27)
	c_8	(1 14 30 5 35 27 21 6)(2 16 17 19)(3 18 34 4 12 31 20 15)(7 9 36 25 10 33 24 28)(8 32 23 11 29 22 13 26)
	c_{12}	(1 35 2 8 7 26)(3 10 28 13 23 34 17 27 30 21 5 12)(4 18 11 20 29 24 36 15 16 14 19 31)(6 9 32 22 25 33)
38	b_{38}	(1 20 4 18 29 38 12 23 37 21 3 32 36 17 6 33 30 10 16 15 28 14 22 7 2 34 19 27 13 26 25 31 11 8 35 24 5 9)
40	b_5	(1 29 26 4 13)(2 28 38 30 22)(3 8 39 16 32)(5 7 27 14 15)(6 25 23 37 9)(10 17 11 40 35)(12 34 18 24 31)(19 36 21 33 20)
	b_{10}	(1 2 3 4 27 20 12 5 6 25)(7 26 17 9 8)(10 13 11 24 33 38 22 31 18 30)(14 34 21 36 37 2315 35 16 29)(19 40 39 32 28)
42	b_{42}	(1 38 15 6 25 14 16 18 7 26 17 35 31 24 12 19 41 4 39 9 40 37 11 3 27 22 30 42 2 10 5 29 21 36 33 23 34 28 32 13 20 8)
44	b_{44}	(1 41 29 9 42 43 8 22 15 26 31 13 2 34 16 21 32 25 39 4 5 38 18 6 3 12 36 10 28 27 14 30 40 24 44 23 7 17 33 20 19 37 11 35)
45	b_8	(1 9 20 37 41 45 34 12)(2 11 8 31 26 5 27 40)(3 39 32 7)(4 6 25 42 36 35 44 28)(13 33 38 19 23 22 21 30)(14 29 43 24 18 17 16 15)
	b_{10}	(1 2 3 4 5 6 7 8 9 10)(11 12 13 14 15 16 17 18 19 20)(21 22 23 24 25 26 27 28 29 30)(31 32 33 34 35 36 37 38 39 40)(41 42 43 44 45)
	c_8	(1 5 16 11 33 13 32 19)(2 15 3 36 12 23 25 40)(4 29 22 41)(6 24 18 37 21 45 34 14)(7 35 28 31 8 30 20 17)(9 10 38 27 42 39 26 44)
	d_8	(1 45 18 31)(2 21 12 34 13 39 22 4)(3 41 35 40)(5 38 10 37 32 16 14 25)(6 9 8 29 26 28 17 27)(7 43 11 19 15 44 36 24)(20 23 42 30)
48	b_{47}	(1 32 48 12 4 43 30 45 20 27 34 9 24 11 2 42 6 22 5 37 23 13 33 38 46 39 19 25 7 18 44 14 28 26 40 10 36 47 29 35 15 8 16 21 41 31 17)
49	b_3	(2 3 5)(4 7 6)(8 35 16)(9 30 20)(10 32 17)(11 34 21)(12 29 18)(13 31 15)(14 33 19)(22 40 46)(23 42 43)(24 37 47)(25 39 44)(26 41 48)(27 36 45)(28 38 49)
	b_6	(2 4 3 7 5 6)(8 45 12 43 13 46)(9 48 14 49 10 44)(11 47)(15 40 16 36 18 42)(17 39 20 41 19 38)(21 37)(22 35 27 29 23 31)(24 34)(25 30 26 33 28 32)
	b_8	(2 16 11 19 7 42 47 39)(3 31 21 30 6 27 37 28)(4 46 24 48 5 12 34 10)(8 23 33 44 43 35 25 14)(9 38 36 13 49 20 15 45)(17 26 22 18 41 32 29 40)
	b_{12}	(2 12 4 27 3 16 7 46 5 31 6 42)(8 23 22 18 15 45 43 35 29 40 36 13)(9 34 25 37 17 11 49 24 33 21 41 47)(10 38 28 14 19 26 48 20 30 44 39 32)
	b_{14}	(1 7 49 48 41 40 33 32 25 24 17 16 9 8)(2 14 43 6 42 47 34 39 26 31 18 23 10 15)(3 21 44 13 36 5 35 46 27 38 19 30 11 22)(4 28 45 20 37 12 29)
	b_{24}	(1 33 14 23 38 48 15 32 7 26 10 44 36 34 21 25 3 47 8 30 42 27 17 46)(2 40 13 16 39 6 19 11)(4 5 12 9 37 41 20 18)(22 31 49)(24 45 43 29 35 28)
	b_{42}	(1 6 14 2 23 41 16 21 22 17 38 7 31 29 37 32 4 15 46 44 3 47 19 30 12 10 18 13 34 45 27 25 33 28 49 11 42 40 48 36 8 26)(5 39 24 9 43 35 20)
	b_{48}	(1 3 42 34 24 31 46 15 33 29 14 23 36 22 48 5 25 26 21 38 12 40 44 32 41 39 7 8 18 11 45 27 9 13 35 19 6 20 43 37 17 16 28 4 30 2 47 10)
	c_6	(2 4 3 7 5 6)(8 43)(9 46 10 49 12 48)(11 45 14 47 13 44)(15 36)(16 39 17 42 19 41)(18 38 21 40 20 37)(22 29)(23 32 24 35 26 34)(25 31 28 33 27 30)
	d_6	(2 4 3 7 5 6)(8 36 29 43 15 22)(9 39 31 49 19 27)(10 42 33 48 16 25)(11 38 35 47 20 23)(12 41 30 46 17 28)(13 37 32 45 21 26)(14 40 34 44 18 24)
50	b_8	(1 27 15 19)(2 37 40 43 34 33 32 14)(3 38)(4 7 16 25 28 18 35 49)(5 22 6 23 45 36 30 47)(8 48 17 9)(10 41 11 42 24 46 31 26)(12 21 13 50 20 44 29 39)
	b_{14}	(1 26 2 27 3 28 4 22 5 23 6 24 7 25)(8 12 9 13 10 14 11)(15 33 16 34 17 35 18 29 19 30 20 31 21 32)(36 43 37 44 38 45 39 46 40 47 41 48 42 49)
	b_{16}	(1 6 11 4 7 8 38 39 10 44 24 37 36 45 26 35)(2 21 19 47 18 20 22 28 49 32 9 23 25 48 3 30)(5 17 46 15 27 40 12 13 16 31 34 14 50 29 41 43)(33 42)
	b_{50}	(1 34 30 17 7 10 44 37 41 18 39 25 26 28 40 20 43 8 32 31 16 47 29 48 14 49 24 12 11 35 46 27 3 19 21 15 4 22 2 6 45 33 36 23 13 9 42 5 50 38)

8.5 Simple Groups

8.64 Remark The relatively recent classification of all finite simple groups has been a major accomplishment of modern group theory. As a result of a remarkable effort by mathematicians from all continents, spanning about two and a half decades (1955-1980), all finite simple groups have been classified. See [932] for an introduction to the proof.

Apart from the cyclic groups $\mathbb{Z}_p$ of prime order, and the alternating groups $\mathcal{A}_n$, there are 16 infinite classes of simple groups of *Lie* type, and an additional 26 *sporadic simple groups* that do not fall in any of the other classes. The simple groups of Lie type consist of 6 classes of so called *classical* groups, and another 10 classes of so called *exceptional simple groups*. The classical simple groups consist of *linear, symplectic, orthogonal* and *unitary* groups. The smallest sporadic simple group is the Mathieu group M_{11} of order 7920, and the largest is a group denoted by F_1, of order $2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$. F_1 is the *Fischer-Griess* group or the (friendly) *giant*. Its existence was established by Griess, who based his construction on results by several other people notably on work of S. Norton, but also of Conway, Fischer, Livingstone, Thorne, and Smith, and his own earlier work. The uniqueness of a simple group of "type F_1" [931] was established by Thompson and Norton. This remarkable giant is constructed as a group of complex matrices of degree $196,883$. It is worth noting that every finite simple group has a representation as a primitive permutation group of rank at most 8, where the stabilizer of a point is a (smaller) simple group.

8.65 Remark Data related to the simple groups of order less than 100 million are displayed in seven columns. (The 106 groups $L_2(q)$ for $q > 5^3$ are omitted; their structure is known and described fully by Dickson [700].) The first column gives the name of the simple group using the conventions in §8.3. Column 2 displays the order of the group. The third column, labeled $\Omega.\{\sigma,\mu\}$ gives information in the form $n.x$. Here, n is a positive integer indicating the order of the outer automorphism group of G, and x is one of the two Greek letters σ,μ. If a μ appears then the group is *minimal simple*, i.e., the group is simple with all of its subgroups solvable. If a σ appears, then the simple group has nonsolvable subgroups, some of which are listed in column 6. In a few instances, as in the case of $L_3(7)$ and $U_4(3)$, in position n of column 3 the isomorphism type of the outer-automorphism group is listed instead of its order. The next column gives information about the conjugacy classes of the group. Only a *signature* which describes the number of conjugacy classes of nonidentity elements of a given order is given. As an example, for group $U_3(3)$ the signature $2^1 3^2 4^3 6^1 7^2 8^2 12^2$ says that there is a single conjugacy class of involutions, two classes of elements of order 3, three classes of elements of order 4, and so on.

Column 5 displays the *primitive spectrum* of the group. This is the list of degrees of the various (inequivalent) primitive permutation representations for the group. An exponent here denotes that there is more than one inequivalent primitive representation of a particular degree for the given group. For example, $\mathcal{A}_6$ has two inequivalent primitive representations of degree 6 (corresponding to two conjugacy classes of $\mathcal{A}_5$'s in $\mathcal{A}_6$), one primitive representation of degree 10, and two inequivalent primitive representations of degree 15. Column 6 displays some nonsolvable subgroups that are contained in the given group. Here the existence of several conjugacy classes of a certain nonsolvable group is indicated by a *multiplicity index* positioned in front of the subgroup specified. For example, the simple group $U_3(5) \cong PSU_3(5^2)$ contains three conjugacy classes of $\mathcal{A}_7$'s, three classes of M_{10}'s, and a class of $\mathcal{S}_5$'s. If a group G is generated as $G = \langle x,y\rangle$, where $|x| = \ell$, $|y| = m$, and $|xy| = n$, G is an (ℓ,m,n) group. The seventh column provides several ways in which G is an (ℓ,m,n) group.

8.66 Table Simple groups of order less than 100,000,000.

Group G	$\|G\|$	$\Omega.\{\sigma,\mu\}$	Conjugacy Classes	Primitive Spec.	n.s. $H < G$	Generated as (ℓ,m,n)
$\mathcal{A}_5 \cong L_2(4) \cong L_2(5)$	60	$2.\mu$	$2^1 3^1 5^2$	$5, 6, 10$	-	(2,3,5)
$L_3(2) \cong L_2(7)$	168	$2.\mu$	$2^1 3^1 4^1 7^2$	$7^2, 8$	-	(2,3,7),(2,4,7),(3,3,4),(2,7,7)
$\mathcal{A}_6 \cong L_2(9)$	360	$4.\sigma$	$2^1 3^2 4^1 5^2$	$6^2, 10, 15^2$	${}_2\mathcal{A}_5$	(2,4,5),(3,3,4),(3,4,5),(2,5,5)
$L_2(8)$	504	$3.\mu$	$2^1 3^1 7^3 9^3$	$9, 28, 36$	-	(2,3,7),(2,3,9),(2,7,9),(2,7,7)
$L_2(11)$	660	$2.\sigma$	$2^1 3^1 5^2 6^1 11^2$	$11^2, 12, 55$	${}_2\mathcal{A}_5$	(2,3,11),(2,5,6),(2,5,11),(2,5,5)
$L_2(13)$	1092	$2.\mu$	$2^1 3^1 6^1 7^3 13^2$	$14, 78, 91^2$	-	(2,3,7),(2,3,13),(2,6,7),(2,7,13)
$L_2(17)$	2448	$2.\mu$	$2^1 3^1 4^1 8^2 9^3 17^2$	$18, 136, 153, 102^2$	-	(2,3,8),(2,3,9),(2,3,17)
$\mathcal{A}_7$	2520	$2.\sigma$	$2^1 3^2 4^1 5^1 6^1 7^2$	$7, 15^2, 21, 35$	$\mathcal{A}_6, \mathcal{A}_5$	(2,4,7),(2,5,7),(2,6,7),(2,7,7)
$L_2(19)$	3420	$2.\sigma$	$2^1 3^1 5^2 9^3 10^2 19^2$	$20, 57^2, 171, 190$	${}_2\mathcal{A}_5$	(2,3,9),(2,3,19),(2,5,9),(2,5,5)
$L_2(16)$	4080	$4.\sigma$	$2^1 3^1 5^2 15^4 17^8$	$17, 68, 120, 136$	$\mathcal{A}_5$	(2,3,17),(2,5,17),(2,3,15),(2,5,15)
$L_3(3)$	5616	$2.\mu$	$2^1 3^2 4^1 6^1 8^2 13^4$	$13^2, 144, 234$	-	(2,3,13),(2,4,13),(2,8,8),(3,4,13)
$U_3(3)$	6048	$2.\sigma$	$2^1 3^2 4^3 6^1 7^2 8^2 12^2$	$28, 36, 63^2$	$L_2(7)$	(2,7,17),(2,6,7),(3,4,7),(3,6,7)
$L_2(23)$	6072	$2.\mu$	$2^1 3^1 4^1 6^1 12^2 11^5 23^2$	$24, 253^3, 276$	-	(2,3,11),(2,3,12),(2,4,6),(2,6,11)
$L_2(25)$	7800	$4.\sigma$	$2^1 3^1 4^1 5^2 6^1 12^2 13^6$	$26, 65^2, 300, 325$	S_5	(2,3,13),(2,3,12),(2,4,13),(2,5,13)
M_{11}	7920	$1.\sigma$	$2^1 3^1 4^1 5^1 6^1 8^2 11^2$	$11, 12, 55, 66, 165$	$L_2(11), \mathcal{S}_5, M_{10}$	(2,5,11),(2,5,8),(2,4,11),(3,4,5)
$L_2(27)$	9828	$6.\mu$	$2^1 3^2 7^3 14^3 13^6$	$28, 351, 378, 819$	-	(2,3,7),(2,3,13),(2,7,13),(2,7,7)
$L_2(29)$	12180	$2.\sigma$	$2^1 3^1 5^2 7^3 14^3 15^4 29^2$	$30, 203^2, 406, 435$	${}_2\mathcal{A}_5$	(2,3,7),(2,5,7),(2,3,15),(2,5,15)
$L_2(31)$	14880	$2.\sigma$	$2^1 3^1 4^1 5^2 8^2 15^4 16^4 31^2$	$32, 248^2, 465, 496$	${}_2\mathcal{A}_5$	(2,3,15),(2,3,8),(2,4,5),(2,5,16)
$\mathcal{A}_8 \cong L_4(2)$	20160	$2.\sigma$	$2^2 3^2 4^2 5^1 6^2 7^2 15^2$	$8, 15^2, 28, 35, 36$	$\mathcal{A}_7, L_2(7), \mathcal{S}_5$	(2,5,7),(2,7,15),(2,4,9),(3,4,15)
$L_3(4) \cong M_{21}$	20160	$12.\sigma$	$2^2 3^1 4^3 5^2 7^2$	$21^2, 56^3, 120^3, 280$	$L_2(7), \mathcal{A}_6, \mathcal{A}_5$	(2,5,7),(2,5,5),(2,4,7),(3,4,7)
$L_2(37)$	25308	$2.\mu$	$2^1 3^1 9^3 6^1 18^3 19^9 37^2$	$38, 666, 703, 2109$	-	(2,3,37),(2,3,9),(2,9,19),(2,18,19)
$U_4(2) \cong S_4(3)$	25920	$2.\sigma$	$2^2 3^4 4^2 5^1 6^6 9^2 12^2$	$27, 36, 40^2, 45$	$\mathcal{S}_5, \mathcal{S}_6$	(2,4,9),(2,5,9),(3,4,9),(3,5,6)
$Sz(8)$	29120	$3.\mu$	$2^1 4^2 5^1 7^3 13^3$	$65, 560, 1456, 2080$	-	(2,5,7),(2,5,13),(2,7,13),(2,4,5)
$L_2(32)$	32736	$5.\mu$	$2^1 3^1 11^5 33^{10} 31^{15}$	$33, 496, 528$	-	(2,3,11),(2,3,31),(2,11,31),(2,31,33)
$L_2(41)$	34440	$2.\mu$	$2^1 3^1 4^1 5^2 7^3 10^2$ $20^4 21^6 41^2$	$42, 820, 861, 1435^2$	${}_2\mathcal{A}_5$	(2,3,7),(2,3,10),(2,5,20),(2,7,20)
$L_2(43)$	39732	$2.\mu$	$2^1 3^1 7^3 11^5 21^6$ $22^5 43^2$	$44, 903, 946, 3311$	-	(2,3,7),(2,3,11),(2,7,11),(3,7,11)
$L_2(47)$	51888	$2.\mu$	$2^1 3^1 4^1 6^1 8^2 12^2$ $23^{11} 24^4 47^2$	$48, 1081, 1128, 2162^2$	-	(2,3,8),(2,3,23),(2,3,47),(3,4,47)

Group G	$\lvert G\rvert$	$\Omega.\{\sigma,\mu\}$	Conjugacy Classes	Primitive Spec.	n.s. $H < G$	Generated as (ℓ,m,n)
$L_2(49)$	58800	$4.\sigma$	$2^1 3^1 4^1 5^2 6^1 7^2$ $8^2 12^2 24^4 25^{10}$	$50, 175^2, 400,$ $1176,\ 1225$	${}_2PGL_2(7)$ ${}_2\mathcal{A}_5$	(2,3,12),(2,7,25), (3,4,5),(3,4,6)
$U_3(4)$	62400	$4.\sigma$	$2^1 3^1 4^1 5^6 10^4 13^4 15^4$	$65, 208, 416, 1600$	$\mathcal{A}_5$	(2,3,13),(2,3,15),(2,10,13),(2,13,15)
$L_2(53)$	74412	$2.\mu$	$2^1 3^1 9^3 13^6 26^6 27^9 53^2$	$54, 1378, 1431, 6201$	-	(2,3,13),(2,3,9),(2,9,26),(2,13,27)
M_{12}	95040	$2.\sigma$	$2^2 3^2 4^2 5^1 6^2 8^2$ $10^1 11^2$	$12^2, 66^2, 144, 220^2,$ $396, 495^2, 1320$	${}_2M_{11},\ {}_2\mathcal{A}_6,$ ${}_3L_2(11),\ \mathcal{S}_5$	(2,3,11),(2,3,10) (2,5,6),(2,5,11)
$L_2(59)$	102660	$2.\sigma$	$2^1 3^1 5^2 6^1 10^2$ $15^4 29^{14} 30^4 59^2$	$60, 1711^3, 1770$	${}_2\mathcal{A}_5$	(2,3,29),(2,3,10), (2,3,15),(3,5,10)
$L_2(61)$	113460	$2.\sigma$	$2^1 3^1 5^2 6^1 10^2$ $15^4 30^4 31^{15} 61^2$	$62, 1891^3, 1830$	${}_2\mathcal{A}_5$	(2,3,31),(2,3,10) (2,5,31),(2,15,31)
$U_3(5)$	126000	$5.\sigma$	$2^1 3^1 4^1 5^4 6^1 7^2 8^2 10^1$	$50^3, 126, 175^3, 525$	${}_3\mathcal{A}_{7,3}\ M_{10}, \mathcal{S}_5$	(2,5,7),(2,4,10),(3,5,7),(3,7,10)
$L_2(67)$	150348	$2.\mu$	$2^1 3^1 11^5 17^8 33^{10} 34^8 67^2$	$68, 2211, 2278, 12529$	-	(2,3,11),(2,3,17),(2,11,67),(2,17,33)
Ja_1	175560	$1.\sigma$	$2^1 3^1 5^2 6^1 7^1$ $10^2 11^1 15^2 19^3$	$266,\ 1045,\ 1463,$ $1540, 1596, 2926, 4180$	$L_2(11),\ \mathcal{A}_5$	(2,3,7),(2,3,11), (2,3,19),(5,7,19)
$L_2(71)$	178920	$2.\sigma$	$2^1 3^1 4^1 5^2 6^1 7^3 9^3$ $12^2 18^3 35^{12} 36^6 71^2$	$72,\ 2485,\ 2556,\ 2982^2$	${}_2\mathcal{A}_5$	(2,3,7),(2,3,71), (2,5,6), (2,6,36),(2,9,35),(5,12,71)
$\mathcal{A}_9$	181440	$2.\sigma$	$2^2 3^3 4^2 5^1 6^2 7^1$ $9^2 10^1 12^1 15^2$	$9, 36, 84, 120^2$ $126,\ 280,\ 840$	$\mathcal{A}_8, \mathcal{S}_7$	(2,3,15),(2,4,7), (3,7,9),(3,7,15)
$L_2(73)$	194472	$2.\mu$	$2^1 3^1 4^1 6^1 9^3$ $12^2 18^3 36^6 37^{18} 73^2$	$74, 2628, 2701,$ 8103^2	-	(2,3,37),(2,3,9), (2,3,73),(2,12,37),(3,6,37)
$L_2(79)$	246480	$2.\sigma$	$2^1 3^1 4^1 5^2 8^2$ $10^2 13^6 20^4 39^{12} 40^8 79^2$	$80, 3081, 3160,$ 4108^2	${}_2\mathcal{A}_5$	(2,3,8),(2,3,13), (2,3,79),(2,13,39),(3,8,13)
$L_2(64)$	262080	$6.\sigma$	$2^1 3^1 5^2 7^3 9^3 13^6 21^6 63^{18} 65^{24}$	$65, 520^2, 2016, 2080$	$\mathcal{A}_5,\ L_2(8)$	(2,3,13),(2,3,63),(2,5,63),(5,13,63)
$L_2(81)$	265680	$4.\sigma$	$2^1 3^2 4^1 5^2 8^2$ $10^2 20^4 40^8 41^{20}$	$82, 369^2, 3240,$ 3321^2	${}_2\mathcal{A}_5$ ${}_2\mathcal{A}_6$	(2,3,41),(2,5,41), (2,10,41),(3,5,8)
$L_2(83)$	285852	$2.\mu$	$2^1 3^1 6^1 7^3 14^3$ $21^6 41^{20} 42^5 83^2$	$84, 3403^2, 3486$ 22821	-	(2,3,7),(2,3,41), (2,41,83),(3,7,41)
$L_2(89)$	352440	$2.\sigma$	$2^1 3^1 4^1 5^2 9^3 11^5$ $15^4 22^5 44^{10} 45^{12} 89^2$	$90, 3916, 4005^2,$ $14685^2, 11748^2$	${}_2\mathcal{A}_5$	(2,3,11),(2,3,9), (2,3,89), (2,11,89),(3,5,45),(3,5,89)
$L_3(5)$	372000	$2.\sigma$	$2^1 3^1 4^3 5^2 6^1 8^2$ $10^1 12^2 20^2 24^4 31^{10}$	$31^2, 3100, 3875, 4000$	$\mathcal{S}_5$	(2,3,31),(2,5,31),(3,8,31), (3,10,24),(3,12,31),(3,20,24)
M_{22}	443520	$2.\sigma$	$2^1 3^1 4^2 5^1 6^1 7^2$ $8^1 11^2$	$22, 77, 176^2, 231,$ $330, 616, 672$	$M_{21},\ {}_2\mathcal{A}_7,$ $L_3(2), \mathcal{A}_6, L_2(11)$	(2,5,7),(2,5,11),(3,4,7), (3,4,11),(3,5,7),(3,5,8)

Group G	$\lvert G\rvert$	$\Omega.\{\sigma,\mu\}$	Conjugacy Classes	Primitive Spec.	n.s. $H<G$	Generated as (ℓ,m,n)
$L_2(97)$	456288	$2.\mu$	$2^1 3^1 4^1 6^1 7^3 8^2 12^2 16^4$ $24^4 48^8 49^{20} 97^2$	$98, 4656, 7453^2,$ 19012^2	-	(2,3,7),(2,3,8),(2,3,49), (3,12,49),(3,7,24),(3,12,49)
$L_2(101)$	515100	$2.\sigma$	$2^1 3^1 5^2 10^2 17^8$ $25^{10} 50^{10} 51^{16} 101^2$	$102, 1717^2, 5050,$ 5151^2	${}_2\mathcal{A}_5$	(2,3,17),(2,3,10), (2,17,25), (3,17,50),(5,17,51),(5,17,101)
$L_2(103)$	546312	$2.\mu$	$2^1 3^1 4^1 13^6 17^8$ $26^6 51^{16} 52^{12} 103^2$	$104, 5253^2,$ $5356, 277823^2$	-	(2,3,13),(2,3,17),(2,51,52), (3,4,52),(4,13,51)
HaJ	604800	$2.\sigma$	$2^2 3^2 4^1 5^4 6^2 7^1$ $8^1 10^4 12^1 15^2$	100,280,315, 525, 840, 1008, 1800, 2016, 10080	$U_3(3), PGL_2(9)$ ${}_4\mathcal{A}_5, L_3(2)$	(2,3,7),(2,3,10),(2,5,7), (2,5,15),(2,6,7),(3,7,10)
$L_2(107)$	612468	$2.\mu$	$2^1 3^1 6^1 9^3 18^3$ $27^9 53^{26} 54^9 107^2$	$108, 5671^2,$ 5778, 51039	-	(2,3,53),(2,3,9), (2,9,54),(3,9,27),(3,9,53)
$L_2(109)$	647460	$2.\mu$	$2^1 3^1 5^2 6^1 9^3 11^5 18^3$ $27^9 54^9 55^{20} 109^2$	110, 5886, $5995^2, 10791^2$	${}_2\mathcal{A}_5$	(2,3,11),(2,3,9),(2,11,55), (2,54,55),(5,6,27),(5,11,18)
$L_2(113)$	721392	$2.\mu$	$2^1 3^1 4^1 7^3 8^2 14^3 19^9$ $28^6 56^{12} 57^{18} 113^2$	114, 6328, $6441^2, 30058^2$	-	(2,3,7),(2,3,19),(2,7,7), (3,7,57),(2,57,57),(3,19,28)
$L_2(11^2)$	885720	$4.\sigma$	$2^1 3^1 4^1 5^2 6^1 10^2 11^2 12^2$ $15^4 20^4 30^4 60^8 61^{30}$	$122, 671^3, 7260,$ $7381^2, 36905^2$	${}_2\mathcal{A}_5$ ${}_2L_2(11)$	(2,3,15),(2,3,61), (2,10,61),(3,5,61),(3,11,61),
$L_2(5^3)$	976500	$6.\sigma$	$2^1 3^1 5^2 7^3 9^3 21^5$ $31^{15} 62^{15} 63^{18}$	126, 7750, $7875^2, 16275$	$\mathcal{A}_5$	(2,3,7),(2,3,31), (2,31,63),(5,7,21)
$S_4(4)$	979200	$4.\sigma$	$2^3 3^2 4^2 5^5 6^2 10^4$ $15^4 17^4$	$85^2, 120^2, 136^2,$ 1360	${}_2\mathcal{A}_5, \mathcal{S}_6,$ ${}_2\mathcal{A}_5 \times \mathcal{A}_5, {}_2L_2(16)$	(2,5,17),(2,4,5), (2,10,15),(3,5,6),(3,10,17)
$S_6(2)$	1451520	$1.\sigma$	$2^4 3^3 4^5 5^1 6^7 7^1$ $8^2 9^1 10^1 12^3 15^1$	28, 36, 63, 120, 135, 315, 336, 960	$U_4(2), U_3(3), \mathcal{S}_8,$ ${}_2\mathcal{S}_6, L_3(2), L_2(8)$	(2,5,7),(2,3,15), (4,5,12),(5,6,7),(6,7,9)
$\mathcal{A}_{10}$	1814400	$2.\sigma$	$2^2 3^3 4^3 5^2 6^3 7^1$ $8^1 9^2 10^1 12^2 15^1$ 21^2	10,45,120,126, 210, 945, 2520	$\mathcal{S}_8, \mathcal{A}_9, \mathcal{A}_7 \times \mathbb{Z}_3,$ $M_{10}, \mathcal{A}_5 \times \mathcal{A}_5,$ $\mathcal{S}_5,\ \mathcal{A}_6 \times \mathcal{A}_4$	(2,3,8),(2,3,9) (3,4,9),(3,7,8) (4,5,9),(6,7,8)
$L_3(7)$	1876896	$\mathcal{S}_3.\sigma$	$2^1 3^1 4^1 6^1 7^4 8^2$ $14^1 16^4 19^6$	$57^2, 5586^3,$ $26068^2, 32928$	$L_2(7)$	(2,3,19),(3,7,8), (3,7,19),(3,8,16)
$U_4(3)$	3265920	$D_8.\sigma$	$2^1 3^4 4^2 5^1 6^3 7^2$ $8^1 9^4 12^1$	$112, 126^2, 162^2, 280, 540,$ $567^2, 1296^4,$ $2835, 4536^2$	${}_3\mathcal{A}_6,\ {}_2U_4(2),$ ${}_2L_3(4),\ U_3(3)$ ${}_4\mathcal{A}_7,\ {}_2M_{10}$	(2,5,7),(3,7,7),(3,7,9), (4,5,6),(4,5,7),(4,5,9), (5,6,7),(6,7,8)
$G_2(3)$	4245696	$2.\sigma$	$2^1 3^5 4^2 6^4 7^1 8^2$ $9^3 12^2 13^2$	$351^2, 364^2, 378^2,$ 2808, 3159, 3888, 7371	${}_2U_3(3),\ {}_2L_3(3),$ $L_2(8),\ L_3(2), L_2(13)$	(2,3,17), (2,4,13)
$S_4(5)$	4680000	$2.\sigma$	$2^2 3^2 4^2 5^6 6^3 10^6$ $12^2 13^3 15^3 20^2 30^2$	$156^2, 300, 325, 4875,$ 6500, 9750, 13000	${}_4\mathcal{A}_5,\ \mathcal{A}_5 \times \mathcal{A}_5,\ L_2(25)$ $\mathcal{A}_6,\ \mathcal{S}_3 \times \mathcal{S}_5$	(2,5,13),(3,10,13),(4,13,15) (3,10,15),(4,5,13),

Group G	$\lvert G\rvert$	$\Omega.\{\sigma,\mu\}$	Conjugacy Classes	Primitive Spec.	n.s. $H<G$	Generated as (ℓ,m,n)
$U_3(8)$	5515776	$12.\sigma$	$2^1 3^3 4^3 6^2 7^3$ $9^3 19^6 21^6$	$513, 3648, 25536^3,$ $34048, 96768$	$L_2(8)$	(2,7,19),(2,7,7),(6,19,19), (19,19,19), (19,19,21),(19,21,21)
$U_3(7)$	5663616	$2.\sigma$	$2^1 3^1 4^3 6^1 7^2 8^{10} 12^2 14^1$ $16^4 24^4 28^2 43^{14} 48^8 56^4$	$344, 2107, 14749,$ $16856, 43904$	${}_2L_2(7)$	(2,3,43),(3,8,43), (3,14,28),(6,8,16),(3,3,43)
$L_4(3)$	6065280	$4.\sigma$	$2^2 3^4 4^3 5^1 6^5 8^1 9^2$ $10^1 12^3 13^4 20^2$	$40^2, 117^2, 130,$ $2106, 8424, 10530$	${}_2L_3(3), {}_2U_4(2)$ $\mathcal{A}_6,\ \mathcal{S}_6$	(2,3,13),(2,13,13),(3,6,13), (3,8,13),(3,13,20),(4,12,13)
$L_5(2)$	9999360	$2.\sigma$	$2^2 3^2 3^3 5^1 6^2 7^2$ $8^1 12^1 14^2 15^2 21^2 31^6$	$31^2, 155^2,$ 64512	${}_2L_4(2),$ ${}_2\mathcal{S}_3 \times L_3(2)$	(2,3,31),(3,14,21),(4,5,6), (4,6,14),(4,7,14),(4,7,31)
M_{23}	10200960	$1.\sigma$	$2^1 3^1 4^1 5^1 6^1 7^2$ $8^1 11^2 14^2 15^2 23^2$	$23, 253^2, 506,$ $1288, 1771, 40320$	$M_{22}, L_3(4), \mathcal{A}_7,$ $\mathcal{A}_8, M_{11}, \mathcal{A}_5$	(2,5,11),(2,5,23),(4,8,14), (4,11,15),(5,6,23),(5,7,11)
$U_5(2)$	13685760	$2.\sigma$	$2^2 3^6 4^3 5^1 6^{14} 8^1$ $9^4 11^2 12^9 15^2 18^2$	$165, 176, 297,$ $1408, 3520, 20736$	$U_4(2),$ ${}_2\mathcal{A}_5, L_2(11)$	(3,9,15),(4,8,9),(5,6,12), (5,6,15),(5,8,11),(6,9,11)
$L_3(8)$	16482816	$6.\sigma$	$2^1 3^1 4^1 7^{11} 9^3$ $14^6 21^6 63^{18} 73^{24}$	$73^2, 5604,$ $75264, 98112$	${}_2L_2(8),$ $L_3(2)$	(3,4,63),(3,7,73) (4,7,63),(4,9,63),(7,9,14)
${}^2F_4(2)'$	17971200	$2.\sigma$	$2^2 3^1 4^3 5^1 6^1 8^4$ $10^1 12^2 13^2 16^4$	$1600^2, 1755, 2304,$ $2925, 12480^2, 14976$	${}_2L_3(3),$ $L_2(25), {}_2\mathcal{A}_6$	(2,3,13)
$\mathcal{A}_{11}$	19958400	$2.\sigma$	$2^2 3^3 4^3 5^2 6^5 7^1 8^1 9^1 10^1$ $11^2 12^3 14^1 15^2 20^1 21^2$	$11, 55, 165, 330,$ $462, 2520^2$	$\mathcal{A}_{10},\ \mathcal{A}_7 \times \mathcal{A}_4, \mathcal{A}_8,$ $\mathcal{S}_9,\ \mathcal{A}_6 \times \mathcal{A}_5, {}_2M_{11}$	(3,4,11),(3,8,11),(4,8,11), (5,8,20),(6,7,9),(6,8,9)
$Sz(32)$	32537600	$5.\mu$	$2^1 4^2 5^1 25^5$ $31^{15} 41^{10}$	$1025, 198400,$ $325376, 524800$	-	(2,4,25),(2,5,31) (2,5,41),(2,31,41)
$L_3(9)$	42456960	$4.\sigma$	$2^1 3^2 4^3 5^2 6^1 7^2$ $8^{10} 10^2 12^2 13^4$ $16^4 20^4 24^4 40^8 80^{16} 91^{24}$	$91^2, 7020, 7560,$ $58968, 110565,$ 155520	$U_3(3),$ $L_3(3)$ $\mathrm{PGL}_2(9),\ {}_2GL_2(9)$	(4,8,24),(4,16,91) (7,8,16),(8,10,12)
$U_3(9)$	42573600	$4.\sigma$	$2^1 3^2 4^1 5^6 6^1 8^2 10^{14} 15^4 16^4$ $20^4 30^4 40^8 73^{24} 80^{16}$	$730, 5913, 59130,$ $70956, 194400$	${}_2\mathcal{A}_6$	(2,5,73),(2,15,10), (2,4,20),(2,40,40)
HS	44352000	$2.\sigma$	$2^2 3^1 4^3 5^3 6^2 7^1$ $8^3 10^2 11^2 12^1$ $15^1 20^2$	$100, 176^2, 1100^2,$ $3850, 4125, 5600^2,$ $5775, 15400, 36960$	$M_{22}, L_3(2), \mathcal{S}_5, \mathcal{A}_6,$ ${}_2U_3(5), L_3(4), \mathcal{A}_5$ $\mathcal{S}_6, \mathcal{S}_8,\ {}_2M_{11}$	(4,5,6),(4,7,11) (5,7,8),(5,7,11) (5,8,10),(6,7,8)
J_3	50232960	$2.\sigma$		$6156, 14688^2, 17442,$ $20520, 23256, 25840,$ $26163, 43605$	$L_2(16),\ {}_2L_2(19)$ $L_2(17), \mathcal{A}_6$ ${}_2\mathcal{A}_5$	(2,3,19)
$U_3(11)$	70915680	$\mathcal{S}_3.\sigma$	$2^1 3^1 4^3 5^2 6^1 8^2$ $10^2 11^4 12^4 20^4$ $22^1 37^1 240^8 44^2$	$1332, 13431, 53724^3,$ $196988^3, 246235,$ $638880, 984940$	${}_4L_2(11),$ ${}_3\mathcal{A}_6$	(4,12,12),(8,37,44) (10,10,37),(37,37,37)

8.6 See Also

§II.5	Gives some Steiner systems with large automorphism groups.
§II.4	t-designs with large automorphism groups.
§III.2	Quasigroups are generalizations of groups.
§VII.5	Groups are used extensively to reduce the computation time in algorithms to produce designs.
[231]	A nice summary treatment of multiply transitive groups.
[930, 1006, 2113]	The fundamental standard references for permutation groups and group actions.

References Cited: [231, 436, 598, 700, 730, 770, 930, 931, 932, 1006, 1441, 1470, 1667, 1801, 2113, 2119]

9 Designs and Matroids

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9.1 Matroids

9.1 A *matroid* is a pair $(E,\mathcal{I})$, where E is a set and $\mathcal{I}$ a non-empty family of subsets of E (*independent sets*) satisfying the conditions

- if $I \in \mathcal{I}$ and $J \subseteq I$, the $J \in \mathcal{I}$;
- (the *Exchange Axiom*) if $I_1, I_2 \in \mathcal{I}$ and $|I_2| > |I_1|$, then there exists $e \in I_2 \setminus I_1$ such that $I_1 \cup \{e\} \in \mathcal{I}$.

9.2 Examples

1. E is the edge set of a graph G; a set of edges is independent if and only if it is a forest. (Such a matroid is a *graphic matroid*.)
2. E is a set of vectors in a vector space V; a set of vectors is independent if and only if it is linear independent. (Such a matroid is a *vector matroid*.)
3. E is a set with a family $(\mathcal{A} = (A_i : i \in I)$ of subsets; a subset of E is independent if and only if it is a partial transversal of $\mathcal{A}$. (A *transversal matroid*.)

9.3 In a matroid $M = (E,\mathcal{I})$,

- a *basis* is a maximal element of $\mathcal{I}$;
- a *circuit* is a minimal element of $\mathcal{P}(E) \setminus \mathcal{I}$;
- the *rank* $\rho(A)$ of a subset A of E is the maximum cardinality of a member of $\mathcal{I}$ contained in A;
- a *flat* is a subset F of E with the property that, for any $e \in E$, $\rho(F \cup \{e\}) = \rho(F)$ implies $e \in F$;
- a *hyperplane* H is a maximal proper flat of M (a flat satisfying $\rho(H) = \rho(E) - 1$).

9.4 Remarks

1. The Exchange Axiom shows that all bases of a matroid have the same cardinality; more generally, all maximal independent subsets of A have rank $\rho(A)$.
2. Matroids can also be defined and axiomatized in terms of their bases, circuits, rank function, flats, or hyperplanes.

3. The set of flats of a matroid is closed under intersection. Also, if F is a flat and x a point not in F, then there is a flat F' with $F \cup \{x\} \in F'$ and $\rho(F') = \rho(F) + 1$.

9.5 A *loop* in a matroid M is an element e such that $\{e\} \notin \mathcal{I}$. Two non-loops e_1, e_2 are *parallel* if $\{e_1, e_2\} \notin \mathcal{I}$. A matroid is *geometric* if it has no loops and no pairs of parallel elements. A geometric matroid is also known as a *combinatorial geometry.*

9.6 Remark From any matroid M, one obtains a geometric matroid (the *geometrization* of M) by deleting loops and identifying parallel elements. The geometrization of a vector space is the corresponding projective space.

9.7 The *truncation* of $M = (E, \mathcal{I})$ to rank r is the matroid on E whose family of independent sets is $\{I \in \mathcal{I} : |I| \le r\}$. Its flats are the flats of M of rank less than r, together with E.

9.8 Example The *free matroid* F_n of rank n is the matroid with $|E| = n$ and $\mathcal{I} = \mathcal{P}(E)$. Its truncation to rank r is the *uniform matroid* $U_{n,r}$.

9.9 Remark For more information on matroids, see Oxley [1698] or Welsh [2104].

9.2 Perfect Matroid Designs

9.10 A *perfect matroid design*, or *PMD*, is a matroid M, of rank r say, for which there exist integers $f_0, f_1, \ldots, f_r$ such that, for $0 \le i \le r$, any flat of rank i has cardinality f_i. The tuple $(f_0, f_1, \ldots, f_r)$ is the *type* of M.

9.11 Remark If M is a PMD of type $(f_0, f_i, \ldots, f_r)$, then the geometrization of M is a PMD of type $(f_0', f_1', \ldots, f_r')$, where $f_i' = (f_i - f_0)/(f_1 - f_0)$. Hence, $f_0' = 0$, $f_1' = 1$.

9.12 Theorem If there exists a PMD of type $(0, 1, f_2, \ldots, f_r)$, then

1. $\displaystyle\prod_{i \le k \le j-1} \frac{f_l - f_k}{f_j - f_k}$ is a non-negative integer for $0 \le i < j \le l \le r$;
2. $f_i - f_{i-1}$ divides $f_{i+1} - f_i$ for $2 \le i \le r-1$;
3. $(f_i - f_{i-1})^2 \le (f_{i+1} - f_i)(f_{i-1} - f_{i-2})$ for $1 \le i \le r-1$.

9.13 Remark The above necessary conditions are not sufficient; for example R. M. Wilson showed that no PMD of type $(0, 1, 3, 7, 43)$ or $(0, 1, 3, 19, 307)$ exists.

9.14 Examples Not many PMDs are known. All known geometric PMDs are truncations of examples on the following list:

1. Free matroids, with $f_i = i$ for all i.
2. Finite projective spaces over a field $\mathbb{F}_q$, with $f_i = (q^i - 1)/(q - 1)$.
3. Finite affine spaces: the points are the vectors in a vector space of rank r over $\mathbb{F}_q$; the independent sets are those which are affine independent. (The set $\{v_1, \ldots, v_k\}$ is *affine independent* if $\{v_2 - v_1, \ldots, v_k - v_1\}$ is linearly independent.) These have $f_i = q^i$.
4. Steiner systems. (Given a Steiner system $S(t, k, v)$ on the set E, take the independent sets to be all sets of cardinality at most t together with all $(t+1)$-sets not contained in a block. Then the hyperplanes are the blocks of S.) These PMDs have rank $t + 1$ and are characterized by the property that $f_i = i$ for $i < t$; also $f_t = k$ and $f_{t+1} = v$.
5. Hall triple systems (§9.3 below): these have rank 4, $f_2 = 3$, and $f_3 = 9$, and the number of points $|E|$ is a power of 3.

9.15 Remark A *line*, resp. *plane* in a PMD is a flat of rank 2, resp. 3. The points and lines of a geometric PMD form a Steiner system $S(2, f_2, f_r)$; the flats form subsystems, so that (for example) any three non-collinear points lie in a unique plane.

9.16 Theorem Let M be a geometric PMD of rank 4.

1. If the planes in M are projective planes (that is, if $f_3 = f_2^2 - f_2 + 1$ with $f_2 > 2$), then M is a truncation of a projective space.
2. If the planes in M are affine planes (that is, if $f_3 = f_2^2$), and if $f_2 > 3$, then M is a truncation of an affine space.

9.17 Remark There is a wide gap between the restrictions imposed by Theorem 9.12 and the known examples. For example, it is not known whether there is a PMD of type $(0, 1, 3, 13, 183)$, $(0, 1, 3, 13, 313)$, or $(0, 1, 3, 15, 183)$.

9.3 Hall Triple Systems and their Algebraic Siblings

9.18 Remark A *triffid* is any PMD of rank 4 with type $(0, 1, 3, 9, 3^n)$. There is an equivalence between those PMDs and each of following structures:

1. *Hall triple system*, i.e. Steiner triple system $S(2, 3, 3^n)$ on E, $|E| = 3^n$, such that for any point $a \in E$ there exist an involution which has a as unique fixed point. (See §VI.28).
2. *Finite exponent 3 commutative Moufang loop* (*exp3-CML*, for short), i.e. a finite commutative loop $(L, \cdot)$, such that for any $x, y, z \in L$, $(x \cdot x)\dot{c}(x \cdot z) = (x \cdot y) \cdot (x \cdot z)$ and $(x \cdot x) \cdot x = 1$ hold.
3. *Distributive Manin quasigroup*, i.e. a groupoid $(Q, \circ)$, such that for any $x, y, z \in Q$, $x \circ y = y \circ x$ and $x \circ (x \circ y) = y$ (i.e, any relation $x \circ y = z$ is preserved under permutation of the variables; in particular, $(Q, \circ)$ is a quasigroup) and all translations are automorphisms.
4. *Restricted Fischer pair* (G, F), i.e. a group G generated by a subset F, such that $x^2 = 1 = (xy)^3$ and $xyx \in F$ for any $x, y \in F$ and such that the commutative center of G is just $\{1\}$.

9.19 Remark The correspondence between the above four definitions is easy to check. For example, a distributive Manin quasigroup arises from a Hall triple system by taking $Q = E$ and defining $x \circ y$, for $x \neq y$, being the unique 3rd point of the triple defined by x, y.

9.20 Remark A triffid is a truncation of an affine space over $\mathbb{F}_3$ if and only if corresponding exp3-CML is a group; then the group is Z_3^n. The first (and smallest) example of a non-associative exp3-CML was given in 1937 by Zassenhaus, by defining, on the set of all 81 $(0, 1, 2)$-sequences $x = (x_1, x_2, x_3, x_4)$, the product as

$$x \cdot y = (x_1 + y_1, x_2 + y_2, x_3 + y_3, x_4 + y_4 + (x_3 - y_3)(x_1 y_2 - x_2 y_1))$$

(all above sums are modulo 3). The number of non-associative exp3-CMLs of order 3^n is $1, 1, 3, 12, > 41$ for $n = 4, 5, 6, 7, 8$, respectively.

9.21 Remark The *dimension* $d(L)$ of a exp3-CML L is the smallest number m such that there exist a $(m + 1)$-set generating it; it is usual dimension of corresponding Steiner triple system. Denote by L_m the *free* exp3-CML of dimension m, i.e. such that any exp3-CML of dimension m is its homeomorhic image. One has $|L_m| = 3^4$, 3^{12}, 3^{49}, 3^{220}, 3^{1028}, 3^{4592} for $m = 3, 4, 5, 6, 7, 8$ (Smith [1915]). Moreover, $d(L) \leq \log_3 |L|$, with equality only if L is associative, i.e. an abelian 3-group. The number of exp3-CMLs L with $|L| = 3^n$ and $d(L) = 4$ is $1, 1, 1, 1, 4$ for $n = 4, 5, 6, 7, 8$ (Bénéteau [187]).

9.22 Remark The associative center $Z(L)$ of a exp3-CML L is the abelian 3-group $\{z \in L : (x \cdot y) \cdot z = x \cdot (y \cdot z)\}$. The *central quotient* $L^{(1)} = L/Z(L)$ is again a exp3-CML; define $L^{(i)}$ as $L^{(i-1)}/Z(L^{(i-1)})$. The *nilpotency class* $k(L)$ of L is the smallest number k such that $L^{(k)}$ is associative. Then $k(L) \leq d(L)$, with equality for the free loop L_m. All exp3-CML's L with $k(L) = 2$ are classified in [1272] and [1813].

9.23 Remark Infinite exp3-CMLs were connected by Manin with cubic hypersurfaces; see, for example [188] for recent developments. For connections with differential geometry (3-webs) and topological algebra, see, for example, [1648].

9.4 Designs in PMDs

9.24 Let t, k, v, λ be integers with $0 \leq t < k < v$ and $\lambda > 0$. Let M be a PMD of rank v. A t-(v, k, λ) *design in* M is a family $\mathcal{B}$ of k-flats of M with the property that every t-flat in M is contained in exactly λ members of $\mathcal{B}$.

9.25 Remarks

1. A t-(v, k, λ) design in the free matroid F_v is just a t-(v, k, λ) design in the usual sense.
2. A t-(v, k, λ) design in a PMD M is also a s-$(v, k\lambda_s)$ design in M for $s < t$, where
$$\lambda_s = \lambda \cdot \prod_{i=s+1}^{t} \frac{f_v - f_s}{f_k - f_s}.$$
3. A t-(v, k, λ) design in a PMD M of type $(f_0, f_1, \ldots, f_v)$ is an ordinary s-design, where $s = \min(t, \max\{i : f_i = i\})$.

9.26 Remark Apart (obviously) from ordinary t-designs, the only case to have been studied is that of t-designs in projective spaces over $\mathbb{F}_q$. To simplify the notation, let $[n]_q = (q^n - 1)/(q - 1)$, so that the projective space has type $([0]_q, [1]_q, \ldots, [v]_q)$; and define the *Gaussian coefficient*
$$\begin{bmatrix} v \\ k \end{bmatrix}_q = \prod_{i=0}^{k-1} \frac{[v-i]_q}{[k-i]_q}$$
to be the number of k-flats. In this case, some theory has been developed. For example, the analogue of the theorem of Ray-Chaudhuri and Wilson (Theorem VI.1.47) holds: in a $2s$-design in M, the number of blocks is at least $\begin{bmatrix} v \\ s \end{bmatrix}_q$.

9.27 Examples

- The first examples were due to Thomas [2007]. They live in $\mathrm{PG}(n, 2)$, where $n+1$ is coprime to 6; blocks are planes, and any line is contained in seven blocks.
- Ray-Chauduhri and Schram [1764] gave a few more constructions.
- A recent investigation appears in Braun *et al.* [328].

9.28 Remark It is unknown whether the analogue of Teirlinck's Theorem holds: do t-designs (without repeated blocks, and such that not every k-flat is a block) exist for all t?

9.5 Permutation groups

9.29 A *base* in a permutation group G is a sequence of points whose pointwise stabiliser in G is the identity. A base is *irredundant* if no point is fixed by the stabilizer of its predecessors.

9.30 Remark In general the bases of a permutation group do not satisfy the matroid basis axioms!

9.31 An *IBIS group* is a permutation group satisfying the following three conditions (shown to be equivalent by Cameron and Fon-Der-Flaass):

- all irredundant bases contain the same number of points;
- the irredundant bases are preserved by reordering;
- the irredundant bases are the bases of a matroid.

The *rank* of an IBIS group is the rank of its associated matroid.

9.32 A permutation group is *base-transitive* if it permutes its ordered bases transitively. Clearly every base-transitive group is an IBIS group; moreover, the associated matroid is a PMD. Define the *type* of the group to be the type of the PMD. A permutation group is base-transitive if and only if the pointwise stabiliser of any sequence of points is transitive on the points it doesn't fix (if any).

9.33 Remark The base-transitive permutation groups with rank greater than 1 have been classified by Maund [1547]: the list is too long to give here. Maund's proof of this theorem used the Classification of Finite Simple Groups (CFSG). An 'elementary' (but by no means easy) determination of base-transitive groups of rank at least 7, not using CFSG, was given by Zil'ber [2186].

9.34 Remark A definition of *permutation geometries*, the analogue in the semilattice of subpermutations (partial bijections) of a set, was given by Cameron and Deza. Following the matroid flat axioms they defined a permutation geometry to be a family $\mathcal{F}$ of subpermutations, closed under intersections, and having the property that if $F \in \mathcal{F}$ and x and y are points with x not in the domain, and y not in the range, of F, there is a unique $F' \in \mathcal{F}$ with rank one greater than the rank of F, extending F and mapping x to y.

A *geometric set* of permutations is a set which consists of the maximal elements in a permutation geometry, and a *geometric group* is a geometric set which forms a group. Now a geometric group is the same thing as a base-transitive group.

9.35 Theorem Let G be an IBIS group of rank $r > 1$ whose associated matroid is uniform. Then G is $(r-1)$-transitive (and the stabilizer of any r points is the identity).

9.36 Remark For $r = 2$, the groups in the conclusion of this theorem are precisely the *Frobenius groups*; a lot of information about their structure is known (Frobenius, Zassenhaus, Thompson). For $r = 3$, they are the *Zassenhaus groups*; these have been determined (Zassenhaus, Feit, Ito, Suzuki) without the use of CFSG. For $r > 3$, they were determined by Gorenstein and Hughes. In particular, the only ones with $r \geq 5$ are symmetric and alternating groups and the Mathieu group M_{12}.

9.6 Some Generalizations of PMD

9.37 Remark A *matroid design* is a matroid whose hyperplanes form a BIBD, i.e. a (b, v, r, k, λ)-configuration on 1-flats. Any PMD is a MD. Any BIBD is a PMD of rank3 if $\lambda = 1$, but never a MD if $\lambda = 2$. Kantor [1246] showed that the point-hyperplanes design of $\mathrm{PG}(n, q)$ is unique *symmetric* BIBD with $\lambda > 1$, which is a MD. Welsh [2104] contain some necessary conditions for a BIBD to be a MD and information of *base designs*, i.e. matroids whose bases are the blocks of a BIBD, and *circuit designs*, i.e. matroids whose circuits are the blocks of a BIBD.

9.38 Remark An *equicardinal matroid* is a matroid, such that all its hyperplanes have the same cardinality k; they are classified for $|E|-k$ being prime or the square of a prime, as well as for the case of rank 3 matroids (Kestenband and Young, 1978).

9.39 **Remark** The following notion generalizes all known PMD of rank 4. A *planar M-space* is a combinatorial geometry of rank 4, such that all its restictions on a 3-flat are isomorphic to the given combinatorial geometry M of rank 3. An example of planar M-space (with at least two 2-flats) having two different 2-flat sizes is known, but only one.

9.40 **Remark** Brylawski [374] considered *latticially uniform* (or latticially homogeneous) matroids, i.e. such that for all i-flats x, all upper intervals $[x, E]$ (or, respectively, all lower intervals $[\emptyset, x]$), in the lattice of the flats) are equal.

9.41 **Remark** PMDs are the extremal case for the families of k-subsets of given v-set intersecting pairwisely in $l_0, l_1 \ldots, l_t$ elements. It was shown in [697] that for $v > v_0(k)$, such family contains at most $\prod_0 \leq i \leq t \frac{v-l_i}{k-l_i}$ sets with equality if and only if it the hyperplane family of a PMD with type $(l_0, l_1, \ldots, l_t, k, v)$.

9.7 See Also

§VI.28	Hall triple systems are perfect matroid designs.
[1698, 2104]	Standard references on matroids.

References Cited: [187, 188, 328, 374, 697, 1246, 1272, 1547, 1648, 1698, 1764, 1813, 1915, 2007, 2104, 2186]

10 Strongly Regular Graphs

Andries E. Brouwer

10.1 Definition and Parameters

10.1 A *strongly regular graph* with parameters (v, k, λ, μ) is a finite graph on v vertices, without loops or multiple edges, regular of degree k (with $0 < k < v-1$, so that there are both edges and nonedges), and such that any two distinct vertices have λ common neighbors when they are adjacent, and μ common neighbors when they are nonadjacent.

10.2 **Remark** The complement of a strongly regular graph with parameters (v, k, λ, μ) is again strongly regular, with parameters $(v, v-k-1, v-2k+\mu-2, v-2k+\lambda)$.

10.3 **Proposition** The 0-1 adjacency matrix A of a strongly regular graph satisfies the equation $A^2 = kI + \lambda A + \mu(J - I - A)$, where I is the identity matrix and J the all-1 matrix, both of size v.

10.4 **Corollary** A has eigenvalues k, r, s with multiplicities 1, f, g, where r, s are found as solutions of $x^2 + (\mu - \lambda)x + (\mu - k) = 0$ (assuming $r \geq 0$, $s \leq -1$) and f, g are found as solutions of $1 + f + g = v$ and $k + fr + gs = 0$.

10.5 A parameter set (v, k, λ, μ) is *feasible* when v, k, λ, μ are nonnegative integers satisfying $\mu(v-k-1) = k(k-1-\lambda)$, $0 < k < v-1$, $v-2k+\mu-2 \geq 0$, $v-2k+\lambda \geq 0$, and f, g are nonnegative integers.

10.6 Proposition If (v, k, λ, μ) is feasible, then either r, s are integral, or there is a t such that $v = 4t + 1$, $k = f = g = 2t + 1$, $\lambda = t - 1$, $\mu = t$, and $r, s = (-1 \pm \sqrt{v})/2$ (the *half case*).

10.7 A strongly regular graph Γ is *imprimitive* when either it or its complement is disconnected.

10.8 Remark If a strongly regular graph Γ with parameters (v, k, λ, μ) is disconnected, then it is the disjoint union of complete graphs, so $v = a(k + 1)$, $\lambda = k - 1$, $\mu = 0$, $r = k$, $s = -1$, $f = a - 1$, $g = ak$ (so that the eigenvalue k has multiplicity a). These graphs exist and are uniquely determined given a and k; they are aK_{k+1}. If the complement of Γ is disconnected, then Γ is the complement of aK_m, that is, it is the complete multipartite graph $K_{a\times m}$ with parameters $v = am$, $k = \mu = (a - 1)m$, $\lambda = (a - 2)m$, $r = 0$, $s = -m$, $f = a(m - 1)$, $g = a - 1$.

10.9 Remark Henceforth Γ is primitive, so that $k > \mu > 0$ and $k > r > 0$.

10.10 Remarks Table 10.12 gives feasible parameter sets for strongly regular graphs on at most 280 vertices. The spectrum, the known constructions, and the known nonexistence results are given. Space limitations have forced the omission of theory and explanations. All information about (semi)partial geometries, gamma and delta spaces, and so on, has been omitted.

10.2 Table of Strongly Regular Graphs

10.11 Remarks The first column of Table 10.12 lists information about existence and enumeration; a minus sign '−' means that there is no such graph; a question mark '?' that no such graph is known; a plus sign '+' that there is at least one such graph; an exclamation mark '!' that there is precisely one such graph, a number n, that at least n such graphs are known; a number n followed by an exclamation mark, that precisely n such graphs exist. Multiplicities are written as exponents. In the comments column, the (or at least some) known constructions are given. It may well happen that several constructions in fact produce the same graph.

Finite fields

Paley(q): For prime powers $q = 4t+1$, the graph with vertex set $\mathbb{F}_q$ where two vertices are adjacent when they differ by a square.

vanLint-Schrijver(u): a graph constructed by the cyclotomic construction in [2065], taking the union of u classes.

Combinatorial

$\binom{n}{2}$ or $T(n)$: the graph on the pairs of an n-set, two pairs being adjacent when they have an element in common. It is the *triangular graph.*

n^2**:** the cartesian product of two complete graphs of size n.

OA(t, n): ($t \geq 3$) the block graph of an orthogonal array OA(t, n) ($t - 2$ MOLS of order n; see §III.3).

$S(2, k, v)$**:** the block graph of a Steiner system with these parameters (see §II.5). More generally, given a quasi-symmetric design (§VI.48), the graph on the blocks, adjacent when they meet in one of the two possible cardinalities, is strongly regular. Much is known, but there is no room for details. An old survey is Neumaier [1660]. From the Witt design $S(5, 8, 24)$ one finds five quasi-symmetric designs: 2-(23,7,21), 2-(22,7,16), 2-(21,7,12), 2-(22,6,5), 2-(21,6,4). Tonchev constructed a quasi-symmetric 2-(45,9,8) design with intersection numbers 1, 3 from the extended binary quadratic residue code of length 48. See also [1150].

Goethals-Seidel(k, r): a graph constructed from a Steiner system $S(2, k, v)$ (with $r = (v-1)/(k-1)$) and a Hadamard matrix of order $r+1$ as in [916]. See also §V.1.

Pasechnik(h): graphs obtained from the tensor product of two nonsymmetric association schemes on h points obtained from a skew Hadamard matrix of order $h+1$; see [1706].

Regular two-graphs

two-graph: the graph belongs to the switching class of a regular two-graph (see §VII.12).

two-graph$-*$: the graph with an isolated point added belongs to the switching class of a regular two-graph. (These are usually comments only, but the existence of a two-graph on $v+1$ points implies the existence of a strongly regular graph on v points.)

RSHCD±: regular two-graphs from regular symmetric Hadamard matrices with constant diagonal (see [354, 916]).

Taylor two-graph for $U_3(q)$: derived from Taylor's regular two-graphs (see [354, 1986, 1987]).

$skewhad^2$**:** derived from the tensor product of the cores of two skew Hadamard matrices of the same order (see [914]).

Projective graphs

$A_{n-1,2}(q)$ or $\left[{n \atop 2}\right]_q$: the graph on the lines in $\mathrm{PG}(n-1, q)$, adjacent when they have a point in common.

$Bilin_{2\times d}(q)$**:** the graph on the $2 \times d$ matrices over $\mathbb{F}_q$, adjacent when their difference has rank 1.

Polar graphs

$O^{\varepsilon}_{2d}(q)$**:** the graph on the isotropic points on a nondegenerate quadric in $\mathrm{PG}(2d-1, q)$, where two points are joined when the connecting line is totally singular.

$O_{2d+1}(q)$**:** such a graph in $\mathrm{PG}(2d, q)$.

$Sp_{2d}(q)$**:** the graph on the points of $\mathrm{PG}(2d-1, q)$ provided with a nondegenerate symplectic form, where two points are joined when the connecting line is totally isotropic.

$U_d(q)$**:** the graph on the isotropic points of $\mathrm{PG}(d-1, q^2)$ provided with a nondegenerate hermitian form, where two points are joined when the connecting line is totally isotropic.

$NO^{\varepsilon}_{2d}(2)$**:** the graph on the nonisotropic points of $\mathrm{PG}(2d-1, 2)$ provided with a nondegenerate quadratic form, where two points are joined when they are orthogonal, i.e., when the connecting line is a tangent.

$NO^{\varepsilon}_{2d}(3)$**:** the graph on one class of nonisotropic points of $\mathrm{PG}(2d-1, 3)$ provided with a nondegenerate quadratic form, where two points are joined when they are orthogonal, i.e., when the connecting line is elliptic.

$NO^{\varepsilon}_{2d+1}(q)$**:** the graph on one class of nondegenerate hyperplanes of $\mathrm{PG}(2d, q)$ provided with a nondegenerate quadratic form, where two hyperplanes are joined when their intersection is degenerate.

$NO^{\varepsilon\perp}_{2d+1}(5)$**:** the graph on one class of nonisotropic points of $\mathrm{PG}(2d, 5)$ provided with a nondegenerate quadratic form, where two points are joined when they are orthogonal.

$NU_n(q)$**:** the graph on the nonisotropic points of $\mathrm{PG}(n-1, q)$ provided with a nondegenerate hermitian form, where two points are joined when the connecting line is a tangent.

Affine polar graphs

$VO^{\varepsilon}_{2d}(q)$**:** the graph on the vectors of a vector space of dimension $2d$ over $\mathbb{F}_q$ provided with a nondegenerate quadratic form Q, where two vectors u and v are joined when $Q(v-u) = 0$.

$VNO^{\varepsilon}_{2d}(q)$**:** ($q$ odd) the graph on the vectors of a vector space of dimension $2d$ over $\mathbb{F}_q$ provided with a nondegenerate quadratic form Q, where two vectors u and v are joined when $Q(v-u)$ is a nonzero square.

Affine difference sets

Let V be an n-dimensional vector space over $\mathbb{F}_q$ and let X be a set of directions (i.e., a subset of the projective space PV). Two vectors are *adjacent* when the line joining them has a direction in X. Then $v = q^n$ and $k = (q-1)|X|$. This graph is strongly regular if and only if there are two integers w_1, w_2 such that all hyperplanes of PV miss either w_1 or w_2 points of X. If this is the case, then $r = k - qw_1$, $s = k - qw_2$ (assuming $w_1 < w_2$), and hence $\mu = k + (k - qw_1)(k - qw_2)$, $\lambda = k - 1 + (k - qw_1 + 1)(k - qw_2 + 1)$. On the other hand, one can view X as the set of columns of the generator matrix of a code, and then the requirement that hyperplanes meet X in one of two cardinalities translates into the requirement that one has a 2-weight code. Thus, projective 2-weight codes, projective sets that have two intersection sizes with hyperplanes, and strongly regular graphs defined by a multiplication invariant difference set on a vector space are equivalent objects. These observations are due to Delsarte [670]. For an extensive discussion, see [415]. One more infinite family is given in Brouwer [349]. There are many sporadic examples — a full discussion is omitted for reasons of space. All constructions 'from a Baer subplane', 'from a hyperoval', 'from a partial spread' etc., refer to this situation, and specify the set X.

Geometric

GQ(s,t): (the collinearity graph of) a generalized quadrangle with $s+1$ points on each line, and $t+1$ lines on each point. See §VI.23 and [1711].

Haemers(q): the graph constructed from a GQ$(q+1, q-1)$ as described in [354], §8A.

Nonexistence: The Half-Case and Conference Matrices

Conf: In the Half Case (parameters $(v,k,\lambda,\mu) = (4t+1, 2t, t-1, t)$ for some integer t), a strongly regular graph exists if and only if a conference matrix C of order $v+1$ exists. Thus v must be a sum of two squares. See §V.6.

Nonexistence: The Krein Conditions A strongly regular graph carries a two-class association scheme, and the standard theory of association schemes applies. In particular, the Krein parameters q^k_{ij} are nonnegative (see [1356, 1844, 1092, 671]). See also §VI.1. Expressed in the parameters, this condition becomes

Krein$_1$**:** $(r+1)(k+r+2rs) \le (k+r)(s+1)^2$.

Krein$_2$**:** $(s+1)(k+s+2rs) \le (k+s)(r+1)^2$.

Nonexistence: The Absolute Bound One has $v \le \frac{1}{2}f(f+3)$ and even $v \le \frac{1}{2}f(f+1)$ if $q^1_{11} \ne 0$ (see Koornwinder [1318]; Delsarte, Goethals and Seidel [674]; Seidel [1857]; Neumaier [1658]). This condition is noted by 'Abs'.

$\mu = 1$**:** If $\mu = 1$, then the set of neighbors of a point is a union of $(\lambda+1)$-cliques, so that $(\lambda+1)|k$. The total number of $(\lambda+2)$-cliques equals $vk/(\lambda+1)(\lambda+2)$ which must be an integer.

10.12 Table Strongly Regular Graphs on at most 280 vertices.

paginate later

$\exists$	v	k	λ	μ	r^f	s^g	Comments
!	5	2	0	1	0.618^2	-1.618^2	pentagon; Paley(5); Seidel 2-graph$-*$
!	9	4	1	2	1^4	-2^4	Paley(9); 3^2; 2-graph$-*$
!	10	3	0	1	1^5	-2^4	Petersen graph [1718]; $NO^-_4(2)$; $NO^{-\perp}_3(5)$; switch $OA(3,2)+*$; 2-graph

$\exists$	v	k	λ	μ	r^f	s^g	Comments
		6	3	4	1^4	-2^5	$\binom{5}{2}$; 2-graph
!	13	6	2	3	1.303^6	-2.303^6	Paley(13); 2-graph$-*$
!	15	6	1	3	1^9	-3^5	$O_5(2)$ polar graph; $Sp_4(2)$ polar graph; $NO_4^-(3)$; 2-graph$-*$
		8	4	4	2^5	-2^9	$\binom{6}{2}$; 2-graph$-*$
!	16	5	0	2	1^{10}	-3^5	$q_{22}^2 = 0$; vanLint-Schrijver(1); $VO_4^-(2)$ affine polar graph; projective binary [5,4] code with weights 2, 4; $RSHCD-$; 2-graph
		10	6	6	2^5	-2^{10}	Clebsch graph [522, 610, 519, 1855]; $q_{11}^1 = 0$; vanLint-Schrijver(2); 2-graph
2!	16	6	2	2	2^6	-2^9	Shrikhande graph [1887]; 4^2; from a partial spread: projective binary [6,4] code with weights 2, 4; $RSHCD+$; 2-graph
		9	4	6	1^9	-3^6	OA(4,3); Bilin$_{2\times 2}$(2); Goethals-Seidel(2,3); $VO_4^+(2)$ affine polar graph; 2-graph
!	17	8	3	4	1.562^8	-2.562^8	Paley(17); 2-graph$-*$
!	21	10	3	6	1^{14}	-4^6	
		10	5	4	3^6	-2^{14}	$\binom{7}{2}$
$-$	21	10	4	5	1.791^{10}	-2.791^{10}	Conf
!	25	8	3	2	3^8	-2^{16}	5^2
		16	9	12	1^{16}	-4^8	OA(5,4)
15!	25	12	5	6	2^{12}	-3^{12}	complete enumeration by Paulus [1707]; Paley(25); OA(5,3); 2-graph$-*$
10!	26	10	3	4	2^{13}	-3^{12}	complete enumeration by Paulus [1707]; switch $OA(5,3)+*$; 2-graph
		15	8	9	2^{12}	-3^{13}	S(2,3,13); 2-graph
!	27	10	1	5	1^{20}	-5^6	$q_{22}^2 = 0$; $O_6^-(2)$ polar graph; $GQ(2,4)$; 2-graph$-*$
		16	10	8	4^6	-2^{20}	Schläfli graph; unique by Seidel [1855]; $q_{11}^1 = 0$; 2-graph$-*$
$-$	28	9	0	4	1^{21}	-5^6	Krein$_2$; Abs
		18	12	10	4^6	-2^{21}	Krein$_1$; Abs
4!	28	12	6	4	4^7	-2^{20}	Chang graphs [452]; $\binom{8}{2}$; 2-graph
		15	6	10	1^{20}	-5^7	$NO_6^+(2)$; Goethals-Seidel(3,3); Taylor 2-graph for $U_3(3)$
41!	29	14	6	7	2.193^{14}	-3.193^{14}	complete enumeration by Bussemaker & Spence [pers.comm.]; Paley(29); 2-graph$-*$
$-$	33	16	7	8	2.372^{16}	-3.372^{16}	Conf
3854!	35	16	6	8	2^{20}	-4^{14}	complete enumeration by McKay & Spence [1565]; 2-graph$-*$
		18	9	9	3^{14}	-3^{20}	S(2,3,15); $A_{3,2}(2)$; $O_6^+(2)$ polar graph; 2-graph$-*$
!	36	10	4	2	4^{10}	-2^{25}	6^2
		25	16	20	1^{25}	-5^{10}	
180!	36	14	4	6	2^{21}	-4^{14}	$U_3(3).2/L_2(7).2$ - subconstituent of Hall-Janko graph; complete enumeration by McKay & Spence [1565]; $RSHCD-$; 2-graph
		21	12	12	3^{14}	-3^{21}	2-graph
!	36	14	7	4	5^8	-2^{27}	$\binom{9}{2}$
		21	10	15	1^{27}	-6^8	
32548!	36	15	6	6	3^{15}	-3^{20}	complete enumeration by McKay & Spence [1565]; OA(6,3); $NO_6^-(2)$; $RSHCD+$; 2-graph
		20	10	12	2^{20}	-4^{15}	$NO_5^-(3)$; 2-graph
+	37	18	8	9	2.541^{18}	-3.541^{18}	Paley(37); 2-graph$-*$
28!	40	12	2	4	2^{24}	-4^{15}	complete enumeration by Spence [1934]; $O_5(3)$ polar graph; $Sp_4(3)$ polar graph
		27	18	18	3^{15}	-3^{24}	$NU(4,2)$

$\exists$	v	k	λ	μ	r^f	s^g	Comments
+	41	20	9	10	2.702^{20}	-3.702^{20}	Paley(41); 2-graph$-*$
78!	45	12	3	3	3^{20}	-3^{24}	complete enumeration by Coolsaet, Degraer & Spence [605]; $U_4(2)$ polar graph
		32	22	24	2^{24}	-4^{20}	$NO_5^+(3)$
!	45	16	8	4	6^9	-2^{35}	$\binom{10}{2}$
		28	15	21	1^{35}	-7^9	
+	45	22	10	11	2.854^{22}	-3.854^{22}	Mathon [1525]; 2-graph$-*$
!	49	12	5	2	5^{12}	-2^{36}	7^2
		36	25	30	1^{36}	-6^{12}	OA(7,6)
−	49	16	3	6	2^{32}	-5^{16}	Bussemaker-Haemers-Mathon-Wilbrink [407]
		32	21	20	4^{16}	-3^{32}	
+	49	18	7	6	4^{18}	-3^{30}	OA(7,3); Pasechnik(7)
		30	17	20	2^{30}	-5^{18}	OA(7,5)
+	49	24	11	12	3^{24}	-4^{24}	Paley(49); OA(7,4); 2-graph$-*$
!	50	7	0	1	2^{28}	-3^{21}	Hoffman-Singleton graph [1112]; $U_3(5^2)$.2/Sym(7)
		42	35	36	2^{21}	-3^{28}	
−	50	21	4	12	1^{42}	-9^7	Abs
		28	18	12	8^7	-2^{42}	Abs
+	50	21	8	9	3^{25}	-4^{24}	switch $OA(7,4)+*$; switch skewhad$^2+*$; 2-graph
		28	15	16	3^{24}	-4^{25}	S(2,4,25); 2-graph
+	53	26	12	13	3.140^{26}	-4.140^{26}	Paley(53); 2-graph$-*$
!	55	18	9	4	7^{10}	-2^{44}	$\binom{11}{2}$
		36	21	28	1^{44}	-8^{10}	
!	56	10	0	2	2^{35}	-4^{20}	Sims-Gewirtz graph [890, 891, 352]; $L_3(4).2^2/\mathrm{Alt}(6).2^2$
		45	36	36	3^{20}	-3^{35}	Witt: intersection 2 graph of a 2-(21,6,4) design with block intersections 0, 2
−	56	22	3	12	1^{48}	-10^7	Krein_2; Abs
		33	22	15	9^7	-2^{48}	Krein_1; Abs
−	57	14	1	4	2^{38}	-5^{18}	Wilbrink-Brouwer [2116]
		42	31	30	4^{18}	-3^{38}	
+	57	24	11	9	5^{18}	-3^{38}	S(2,3,19)
		32	16	20	2^{38}	-6^{18}	
−	57	28	13	14	3.275^{28}	-4.275^{28}	Conf
+	61	30	14	15	3.405^{30}	-4.405^{30}	Paley(61); 2-graph$-*$
−	63	22	1	11	1^{55}	-11^7	Krein_2; Abs
		40	28	20	10^7	-2^{55}	Krein_1; Abs
+	63	30	13	15	3^{35}	-5^{27}	$O_7(2)$ polar graph; $Sp_6(2)$ polar graph; 2-graph$-*$
		32	16	16	4^{27}	-4^{35}	S(2,4,28); $NU(3,3)$; 2-graph$-*$
!	64	14	6	2	6^{14}	-2^{49}	8^2; from a partial spread of 3-spaces: projective binary [14,6] code with weights 4, 8
		49	36	42	1^{49}	-7^{14}	OA(8,7)
167!	64	18	2	6	2^{45}	-6^{18}	complete enumeration by Haemers & Spence [1000]; $GQ(3,5)$; from a hyperoval: projective 4-ary [6,3] code with weights 4, 6
		45	32	30	5^{18}	-3^{45}	
−	64	21	0	10	1^{56}	-11^7	Krein_2; Abs
		42	30	22	10^7	-2^{56}	Krein_1; Abs
+	64	21	8	6	5^{21}	-3^{42}	OA(8,3); $\mathrm{Bilin}_{2\times 3}(2)$; from a Baer subplane: projective 4-ary [7,3] code with weights 4, 6; Brouwer [349]; from a partial spread of 3-spaces: projective binary [21,6] code with weights 8, 12
		42	26	30	2^{42}	-6^{21}	OA(8,6)
+	64	27	10	12	3^{36}	-5^{27}	from a unital: projective 4-ary [9,3] code with weights 6, 8; $VO_6^-(2)$ affine polar graph; $RSHCD-$; 2-graph
		36	20	20	4^{27}	-4^{36}	2-graph

∃	v	k	λ	μ	r^f	s^g	Comments
+	64	28	12	12	4^{28}	-4^{35}	OA(8,4); from a partial spread of 3-spaces: projective binary [28,6] code with weights 12, 16; $RSHCD+$; 2-graph
		35	18	20	3^{35}	-5^{28}	OA(8,5); Goethals-Seidel(2,7); $VO_6^+(2)$ affine polar graph; 2-graph
−	64	30	18	10	10^8	-2^{55}	Abs
		33	12	22	1^{55}	-11^8	Abs
?	65	32	15	16	3.531^{32}	-4.531^{32}	2-graph−∗?
!	66	20	10	4	8^{11}	-2^{54}	$\binom{12}{2}$
		45	28	36	1^{54}	-9^{11}	
?	69	20	7	5	5^{23}	-3^{45}	
		48	32	36	2^{45}	-6^{23}	S(2,6,46)?
−	69	34	16	17	3.653^{34}	-4.653^{34}	Conf
+	70	27	12	9	6^{20}	-3^{49}	S(2,3,21)
		42	23	28	2^{49}	-7^{20}	
+	73	36	17	18	3.772^{36}	-4.772^{36}	Paley(73); 2-graph−∗
?	75	32	10	16	2^{56}	-8^{18}	2-graph−∗?
		42	25	21	7^{18}	-3^{56}	2-graph−∗?
−	76	21	2	7	2^{56}	-7^{19}	Haemers [999]
		54	39	36	6^{19}	-3^{56}	
?	76	30	8	14	2^{57}	-8^{18}	2-graph?
		45	28	24	7^{18}	-3^{57}	2-graph?
?	76	35	18	14	7^{19}	-3^{56}	2-graph?
		40	18	24	2^{56}	-8^{19}	2-graph?
!	77	16	0	4	2^{55}	-6^{21}	S(3,6,22); $M_{22}.2/2^4:$ Sym(6); unique by Brouwer [347]; subconstituent of Higman-Sims graph
		60	47	45	5^{21}	-3^{55}	Witt 3-(22,6,1): intersection 2 graph of a 2-(22,6,5) design with block intersections 0, 2
−	77	38	18	19	3.887^{38}	-4.887^{38}	Conf
!	78	22	11	4	9^{12}	-2^{65}	$\binom{13}{2}$
		55	36	45	1^{65}	-10^{12}	
!	81	16	7	2	7^{16}	-2^{64}	9^2; Brouwer [349]; from a partial spread: projective ternary [8,4] code with weights 3, 6
		64	49	56	1^{64}	-8^{16}	OA(9,8)
!	81	20	1	6	2^{60}	-7^{20}	unique by Brouwer & Haemers [351]; $VO_4^-(3)$ affine polar graph; projective ternary [10,4] code with weights 6, 9
		60	45	42	6^{20}	-3^{60}	
+	81	24	9	6	6^{24}	-3^{56}	OA(9,3); $VNO_4^+(3)$ affine polar graph; from a partial spread: projective ternary [12,4] code with weights 6, 9
		56	37	42	2^{56}	-7^{24}	OA(9,7)
+	81	30	9	12	3^{50}	-6^{30}	$VNO_4^-(3)$ affine polar graph; Hamada-Helleseth [1016]: projective ternary [15,4] code with weights 9, 12
		50	31	30	5^{30}	-4^{50}	
+	81	32	13	12	5^{32}	-4^{48}	OA(9,4); Bilin$_{2\times 2}$(3); $VO_4^+(3)$ affine polar graph; from a partial spread: projective ternary [16,4] code with weights 9, 12
		48	27	30	3^{48}	-6^{32}	OA(9,6)
−	81	40	13	26	1^{72}	-14^8	Abs
		40	25	14	13^8	-2^{72}	Abs
+	81	40	19	20	4^{40}	-5^{40}	Paley(81); OA(9,5); projective ternary [20,4] code with weights 12, 15; 2-graph−∗
+	82	36	15	16	4^{41}	-5^{40}	switch $OA(9,5)+*$; 2-graph

$\exists$	v	k	λ	μ	r^f	s^g	Comments
		45	24	25	4^{40}	-5^{41}	S(2,5,41); 2-graph
?	85	14	3	2	4^{34}	-3^{50}	
		70	57	60	2^{50}	-5^{34}	
+	85	20	3	5	3^{50}	-5^{34}	$O_5(4)$ polar graph; $Sp_4(4)$ polar graph
		64	48	48	4^{34}	-4^{50}	
?	85	30	11	10	5^{34}	-4^{50}	
		54	33	36	3^{50}	-6^{34}	S(2,6,51)?
?	85	42	20	21	4.110^{42}	-5.110^{42}	2-graph$-*$?
?	88	27	6	9	3^{55}	-6^{32}	
		60	41	40	5^{32}	-4^{55}	
+	89	44	21	22	4.217^{44}	-5.217^{44}	Paley(89); 2-graph$-*$
!	91	24	12	4	10^{13}	-2^{77}	$\binom{14}{2}$
		66	45	55	1^{77}	-11^{13}	
−	93	46	22	23	4.322^{46}	-5.322^{46}	Conf
?	95	40	12	20	2^{75}	-10^{19}	2-graph$-*$?
		54	33	27	9^{19}	-3^{75}	2-graph$-*$?
+	96	19	2	4	3^{57}	-5^{38}	Haemers(4)
		76	60	60	4^{38}	-4^{57}	
+	96	20	4	4	4^{45}	-4^{50}	$GQ(5,3)$
		75	58	60	3^{50}	-5^{45}	
?	96	35	10	14	3^{63}	-7^{32}	
		60	38	36	6^{32}	-4^{63}	
?	96	38	10	18	2^{76}	-10^{19}	2-graph?
		57	36	30	9^{19}	-3^{76}	2-graph?
?	96	45	24	18	9^{20}	-3^{75}	2-graph?
		50	22	30	2^{75}	-10^{20}	2-graph?
+	97	48	23	24	4.424^{48}	-5.424^{48}	Paley(97); 2-graph$-*$
?	99	14	1	2	3^{54}	-4^{44}	
		84	71	72	3^{44}	-4^{54}	
?	99	42	21	15	9^{21}	-3^{77}	
		56	28	36	2^{77}	-10^{21}	
+	99	48	22	24	4^{54}	-6^{44}	2-graph$-*$
		50	25	25	5^{44}	-5^{54}	S(2,5,45); 2-graph$-*$
!	100	18	8	2	8^{18}	-2^{81}	10^2
		81	64	72	1^{81}	-9^{18}	
!	100	22	0	6	2^{77}	-8^{22}	Higman-Sims graph [1094]; $HS.2/M_{22}.2$; unique by Gewirtz [890]; $q^2_{22} = 0$
		77	60	56	7^{22}	-3^{77}	$q^1_{11} = 0$
+	100	27	10	6	7^{27}	-3^{72}	OA(10,3)
		72	50	56	2^{72}	-8^{27}	OA(10,8)?
?	100	33	8	12	3^{66}	-7^{33}	
		66	44	42	6^{33}	-4^{66}	
+	100	33	14	9	8^{24}	-3^{75}	S(2,3,25)
		66	41	48	2^{75}	-9^{24}	
−	100	33	18	7	13^{11}	-2^{88}	Abs
		66	39	52	1^{88}	-14^{11}	Abs
+	100	36	14	12	6^{36}	-4^{63}	Hall-Janko graph; $J_2.2/U_3(3).2$; subconstituent of $G_2(4)$ graph; OA(10,4)
		63	38	42	3^{63}	-7^{36}	OA(10,7)?
+	100	44	18	20	4^{55}	-6^{44}	Jørgensen-Klin graph [1212]; $RSHCD-$; 2-graph
		55	30	30	5^{44}	-5^{55}	2-graph
+	100	45	20	20	5^{45}	-5^{54}	OA(10,5)?; $RSHCD+$; 2-graph
		54	28	30	4^{54}	-6^{45}	OA(10,6)?; 2-graph
+	101	50	24	25	4.525^{50}	-5.525^{50}	Paley(101); 2-graph$-*$
!	105	26	13	4	11^{14}	-2^{90}	$\binom{15}{2}$

∃	v	k	λ	μ	r^f	s^g	Comments
		78	55	66	1^{90}	-12^{14}	
!	105	32	4	12	2^{84}	-10^{20}	Aut$L_3(4)$ on flags (rk 4) [Goethals & Seidel], unique by Coolsaet-Degraer [604]
		72	51	45	9^{20}	-3^{84}	
?	105	40	15	15	5^{48}	-5^{56}	
		64	38	40	4^{56}	-6^{48}	
?	105	52	21	30	2^{84}	-11^{20}	
		52	29	22	10^{20}	-3^{84}	
−	105	52	25	26	4.623^{52}	-5.623^{52}	Conf
+	109	54	26	27	4.720^{54}	-5.720^{54}	Paley(109); 2-graph−∗
?	111	30	5	9	3^{74}	-7^{36}	
		80	58	56	6^{36}	-4^{74}	
+	111	44	19	16	7^{36}	-4^{74}	S(2,4,37)
		66	37	42	3^{74}	-8^{36}	
!	112	30	2	10	2^{90}	-10^{21}	unique by Cameron, Goethals & Seidel [423]; subconstituent of McLaughlin graph; $q_{22}^2 = 0$; $O_6^-(3)$ polar graph
		81	60	54	9^{21}	-3^{90}	$q_{11}^1 = 0$
?	112	36	10	12	4^{63}	-6^{48}	
		75	50	50	5^{48}	-5^{63}	
+	113	56	27	28	4.815^{56}	-5.815^{56}	Paley(113); 2-graph−∗
?	115	18	1	3	3^{69}	-5^{45}	
		96	80	80	4^{45}	-4^{69}	
+	117	36	15	9	9^{26}	-3^{90}	Wallis; S(2,3,27); $NO_3^+(3)$; Lines in $AG(3,3)$ (rk 4)
		80	52	60	2^{90}	-10^{26}	
?	117	58	28	29	4.908^{58}	-5.908^{58}	2-graph−∗?
+	119	54	21	27	3^{84}	-9^{34}	$O_8^-(2)$ polar graph; 2-graph−∗
		64	36	32	8^{34}	-4^{84}	2-graph−∗
!	120	28	14	4	12^{15}	-2^{104}	$\binom{16}{2}$
		91	66	78	1^{104}	-13^{15}	
?	120	34	8	10	4^{68}	-6^{51}	
		85	60	60	5^{51}	-5^{68}	
?	120	35	10	10	5^{56}	-5^{63}	
		84	58	60	4^{63}	-6^{56}	
!	120	42	8	18	2^{99}	-12^{20}	$L_3(4)$ on Baer subplanes (rk 5), unique by Coolsaet-Degraer [604]
		77	52	44	11^{20}	-3^{99}	Witt: intersection 3 graph of a 2-(21,7,12) design with block intersections 1, 3
+	120	51	18	24	3^{85}	-9^{34}	$NO_5^-(4)$; 2-graph
		68	40	36	8^{34}	-4^{85}	2-graph
+	120	56	28	24	8^{35}	-4^{84}	2-graph
		63	30	36	3^{84}	-9^{35}	dist. 2 in J(10,3) - Mathon; $NO_8^+(2)$; Goethals-Seidel(3,7); 2-graph
!	121	20	9	2	9^{20}	-2^{100}	11^2
		100	81	90	1^{100}	-10^{20}	OA(11,10)
+	121	30	11	6	8^{30}	-3^{90}	OA(11,3)
		90	65	72	2^{90}	-9^{30}	OA(11,9)
?	121	36	7	12	3^{84}	-8^{36}	
		84	59	56	7^{36}	-4^{84}	
+	121	40	15	12	7^{40}	-4^{80}	OA(11,4)
		80	51	56	3^{80}	-8^{40}	OA(11,8)
?	121	48	17	20	4^{72}	-7^{48}	
		72	43	42	6^{48}	-5^{72}	
+	121	50	21	20	6^{50}	-5^{70}	OA(11,5); Pasechnik(11)
		70	39	42	4^{70}	-7^{50}	OA(11,7)
−	121	56	15	35	1^{112}	-21^{8}	Abs

$\exists$	v	k	λ	μ	r^f	s^g	Comments
		64	42	24	20^{8}	-2^{112}	Abs
+	121	60	29	30	5^{60}	-6^{60}	Paley(121); $q_{11}^1 = 0$; OA(11,6); 2-graph$-*$
+	122	55	24	25	5^{61}	-6^{60}	switch $OA(11,6)+*$; switch skewhad$^2 + *$; 2-graph
		66	35	36	5^{60}	-6^{61}	S(2,6,61)?; 2-graph
+	125	28	3	7	3^{84}	-7^{40}	$GQ(4,6)$
		96	74	72	6^{40}	-4^{84}	
−	125	48	28	12	18^{10}	-2^{114}	Abs
		76	39	57	1^{114}	-19^{10}	Abs
+	125	52	15	26	2^{104}	-13^{20}	2-graph$-*$
		72	45	36	12^{20}	-3^{104}	2-graph$-*$
+	125	62	30	31	5.090^{62}	-6.090^{62}	Paley(125); 2-graph$-*$
+	126	25	8	4	7^{35}	-3^{90}	dist. 1 or 4 in J(9,4) - Mathon, Buekenhout & Hubaut
		100	78	84	2^{90}	-8^{35}	
+	126	45	12	18	3^{90}	-9^{35}	$NO_6^-(3)$
		80	52	48	8^{35}	-4^{90}	
+	126	50	13	24	2^{105}	-13^{20}	Goethals; 2-graph
		75	48	39	12^{20}	-3^{105}	2-graph
+	126	60	33	24	12^{21}	-3^{104}	2-graph
		65	28	39	2^{104}	-13^{21}	Taylor 2-graph for $U_3(5)$
−	129	64	31	32	5.179^{64}	-6.179^{64}	Conf
+	130	48	20	16	8^{39}	-4^{90}	S(2,4,40); $A_{3,2}(3)$; $O_6^+(3)$ polar graph
		81	48	54	3^{90}	-9^{39}	
?	133	24	5	4	5^{56}	-4^{76}	
		108	87	90	3^{76}	-6^{56}	
?	133	32	6	8	4^{76}	-6^{56}	
		100	75	75	5^{56}	-5^{76}	
?	133	44	15	14	6^{56}	-5^{76}	
		88	57	60	4^{76}	-7^{56}	
−	133	66	32	33	5.266^{66}	-6.266^{66}	Conf
+	135	64	28	32	4^{84}	-8^{50}	2-graph$-*$
		70	37	35	7^{50}	-5^{84}	$O_8^+(2)$ polar graph; 2-graph$-*$
?	136	30	8	6	6^{51}	-4^{84}	
		105	80	84	3^{84}	-7^{51}	
!	136	30	15	4	13^{16}	-2^{119}	$\binom{17}{2}$
		105	78	91	1^{119}	-14^{16}	
+	136	60	24	28	4^{85}	-8^{50}	2-graph
		75	42	40	7^{50}	-5^{85}	$NO_5^+(4)$; 2-graph
+	136	63	30	28	7^{51}	-5^{84}	$NO_8^-(2)$; 2-graph
		72	36	40	4^{84}	-8^{51}	2-graph
+	137	68	33	34	5.352^{68}	-6.352^{68}	Paley(137); 2-graph$-*$
−	141	70	34	35	5.437^{70}	-6.437^{70}	Conf
+	143	70	33	35	5^{77}	-7^{65}	2-graph$-*$
		72	36	36	6^{65}	-6^{77}	S(2,6,66); 2-graph$-*$
!	144	22	10	2	10^{22}	-2^{121}	12^2
		121	100	110	1^{121}	-11^{22}	OA(12,11)?
+	144	33	12	6	9^{33}	-3^{110}	OA(12,3)
		110	82	90	2^{110}	-10^{33}	OA(12,10)?
+	144	39	6	12	3^{104}	-9^{39}	$L_3(3)$ (rk 8)
		104	76	72	8^{39}	-4^{104}	
+	144	44	16	12	8^{44}	-4^{99}	OA(12,4)
		99	66	72	3^{99}	-9^{44}	OA(12,9)?
?	144	52	16	20	4^{91}	-8^{52}	
		91	58	56	7^{52}	-5^{91}	
+	144	55	22	20	7^{55}	-5^{88}	OA(12,5)
		88	52	56	4^{88}	-8^{55}	OA(12,8)?

$\exists$	v	k	λ	μ	r^f	s^g	Comments
−	144	65	16	40	1^{135}	-25^{8}	Krein$_2$; Abs
		78	52	30	24^{8}	-2^{135}	Krein$_1$; Abs
+	144	65	28	30	5^{78}	-7^{65}	$RSHCD-$; 2-graph
		78	42	42	6^{65}	-6^{78}	2-graph
+	144	66	30	30	6^{66}	-6^{77}	OA(12,6); $RSHCD+$; 2-graph
		77	40	42	5^{77}	-7^{66}	OA(12,7); Goethals-Seidel(2,11); 2-graph
?	145	72	35	36	5.521^{72}	-6.521^{72}	2-graph−∗?
?	147	66	25	33	3^{110}	-11^{36}	2-graph−∗?
		80	46	40	10^{36}	-4^{110}	2-graph−∗?
?	148	63	22	30	3^{111}	-11^{36}	2-graph?
		84	50	44	10^{36}	-4^{111}	2-graph?
?	148	70	36	30	10^{37}	-4^{110}	2-graph?
		77	36	44	3^{110}	-11^{37}	2-graph?
+	149	74	36	37	5.603^{74}	-6.603^{74}	Paley(149); 2-graph−∗
!	153	32	16	4	14^{17}	-2^{135}	$\binom{18}{2}$
		120	91	105	1^{135}	-15^{17}	
?	153	56	19	21	5^{84}	-7^{68}	
		96	60	60	6^{68}	-6^{84}	
?	153	76	37	38	5.685^{76}	-6.685^{76}	2-graph−∗?
?	154	48	12	16	4^{98}	-8^{55}	
		105	72	70	7^{55}	-5^{98}	
−	154	51	8	21	2^{132}	-15^{21}	Krein$_2$
		102	71	60	14^{21}	-3^{132}	Krein$_1$
?	154	72	26	40	2^{132}	-16^{21}	
		81	48	36	15^{21}	-3^{132}	
+	155	42	17	9	11^{30}	-3^{124}	S(2,3,31); $A_{4,2}(2)$
		112	78	88	2^{124}	-12^{30}	
+	156	30	4	6	4^{90}	-6^{65}	$O_5(5)$ polar graph; $Sp_4(5)$ polar graph
		125	100	100	5^{65}	-5^{90}	
+	157	78	38	39	5.765^{78}	-6.765^{78}	Paley(157); 2-graph−∗
?	160	54	18	18	6^{75}	-6^{84}	
		105	68	70	5^{84}	-7^{75}	
−	161	80	39	40	5.844^{80}	-6.844^{80}	Conf
?	162	21	0	3	3^{105}	-6^{56}	
		140	121	120	5^{56}	-4^{105}	
?	162	23	4	3	5^{69}	-4^{92}	
		138	117	120	3^{92}	-6^{69}	
?	162	49	16	14	7^{63}	-5^{98}	
		112	76	80	4^{98}	-8^{63}	
!	162	56	10	24	2^{140}	-16^{21}	$q^2_{22}=0$
		105	72	60	15^{21}	-3^{140}	$U_4(3)/L_3(4)$; unique by Cameron, Goethals & Seidel [423]; subconstituent of McLaughlin graph; $q^1_{11}=0$
?	162	69	36	24	15^{23}	-3^{138}	
		92	46	60	2^{138}	-16^{23}	
+	165	36	3	9	3^{120}	-9^{44}	$U_5(2)$ polar graph
		128	100	96	8^{44}	-4^{120}	
−	165	82	40	41	5.923^{82}	-6.923^{82}	Conf
!	169	24	11	2	11^{24}	-2^{144}	13^2
		144	121	132	1^{144}	-12^{24}	OA(13,12)
+	169	36	13	6	10^{36}	-3^{132}	OA(13,3)
		132	101	110	2^{132}	-11^{36}	OA(13,11)
?	169	42	5	12	3^{126}	-10^{42}	
		126	95	90	9^{42}	-4^{126}	
+	169	48	17	12	9^{48}	-4^{120}	OA(13,4)
		120	83	90	3^{120}	-10^{48}	OA(13,10)

∃	v	k	λ	μ	r^f	s^g	Comments
?	169	56	15	20	4^{112}	-9^{56}	
		112	75	72	8^{56}	-5^{112}	
+	169	60	23	20	8^{60}	-5^{108}	OA(13,5)
		108	67	72	4^{108}	-9^{60}	OA(13,9)
?	169	70	27	30	5^{98}	-8^{70}	
		98	57	56	7^{70}	-6^{98}	
+	169	72	31	30	7^{72}	-6^{96}	OA(13,6)
		96	53	56	5^{96}	-8^{72}	OA(13,8)
+	169	84	41	42	6^{84}	-7^{84}	Paley(169); OA(13,7); 2-graph−∗
+	170	78	35	36	6^{85}	-7^{84}	switch $OA(13,7)+*$; 2-graph
		91	48	49	6^{84}	-7^{85}	S(2,7,85)?; 2-graph
!	171	34	17	4	15^{18}	-2^{152}	$\binom{19}{2}$
		136	105	120	1^{152}	-16^{18}	
?	171	50	13	15	5^{95}	-7^{75}	
		120	84	84	6^{75}	-6^{95}	
?	171	60	15	24	3^{132}	-12^{38}	
		110	73	66	11^{38}	-4^{132}	
+	173	86	42	43	6.076^{86}	-7.076^{86}	Paley(173); 2-graph−∗
+	175	30	5	5	5^{84}	-5^{90}	$GQ(6,4)$
		144	118	120	4^{90}	-6^{84}	
?	175	66	29	22	11^{42}	-4^{132}	
		108	63	72	3^{132}	-12^{42}	
+	175	72	20	36	2^{153}	-18^{21}	edges of Hoffman-Singleton graph; 2-graph−∗
		102	65	51	17^{21}	-3^{153}	2-graph−∗
?	176	25	0	4	3^{120}	-7^{55}	
		150	128	126	6^{55}	-4^{120}	
+	176	40	12	8	8^{55}	-4^{120}	
		135	102	108	3^{120}	-9^{55}	$NU(5,2)$
+	176	45	18	9	12^{32}	-3^{143}	S(2,3,33)
		130	93	104	2^{143}	-13^{32}	
+	176	49	12	14	5^{98}	-7^{77}	Higman symmetric 2-design [1095, 1918, 1904, 345]
		126	90	90	6^{77}	-6^{98}	
!	176	70	18	34	2^{154}	-18^{21}	$S(4,7,23) \setminus S(3,6,22)$; $M_{22}/\mathrm{Alt}(7)$; unique by Coolsaet-Degraer [604]; 2-graph
		105	68	54	17^{21}	-3^{154}	Witt 3-(22,7,4): intersection 3 graph of a 2-(22,7,16) design with block intersections 1, 3; 2-graph
?	176	70	24	30	4^{120}	-10^{55}	
		105	64	60	9^{55}	-5^{120}	
−	176	70	42	18	26^{10}	-2^{165}	Abs
		105	52	78	1^{165}	-27^{10}	Abs
+	176	85	48	34	17^{22}	-3^{153}	Haemers; 2-graph
		90	38	54	2^{153}	-18^{22}	2-graph
−	177	88	43	44	6.152^{88}	-7.152^{88}	Conf
+	181	90	44	45	6.227^{90}	-7.227^{90}	Paley(181); 2-graph−∗
?	183	52	11	16	4^{122}	-9^{60}	
		130	93	90	8^{60}	-5^{122}	
+	183	70	29	25	9^{60}	-5^{122}	S(2,5,61)
		112	66	72	4^{122}	-10^{60}	
−	184	48	2	16	2^{160}	-16^{23}	Krein_2
		135	102	90	15^{23}	-3^{160}	Krein_1
?	185	92	45	46	6.301^{92}	-7.301^{92}	2-graph−∗?
?	189	48	12	12	6^{90}	-6^{98}	
		140	103	105	5^{98}	-7^{90}	
?	189	60	27	15	15^{28}	-3^{160}	
		128	82	96	2^{160}	-16^{28}	
?	189	88	37	44	4^{132}	-11^{56}	2-graph−∗?

$\exists$	v	k	λ	μ	r^f	s^g	Comments
		100	55	50	10^{56}	-5^{132}	2-graph$-*$?
$-$	189	94	46	47	6.374^{94}	-7.374^{94}	Conf
!	190	36	18	4	16^{19}	-2^{170}	$\binom{20}{2}$
		153	120	136	1^{170}	-17^{19}	
?	190	45	12	10	7^{75}	-5^{114}	
		144	108	112	4^{114}	-8^{75}	
?	190	84	33	40	4^{133}	-11^{56}	2-graph?
		105	60	55	10^{56}	-5^{133}	2-graph?
+	190	84	38	36	8^{75}	-6^{114}	S(2,6,76)
		105	56	60	5^{114}	-9^{75}	
?	190	90	45	40	10^{57}	-5^{132}	2-graph?
		99	48	55	4^{132}	-11^{57}	2-graph?
+	193	96	47	48	6.446^{96}	-7.446^{96}	Paley(193); 2-graph$-*$
+	195	96	46	48	6^{104}	-8^{90}	2-graph$-*$
		98	49	49	7^{90}	-7^{104}	S(2,7,91); 2-graph$-*$
!	196	26	12	2	12^{26}	-2^{169}	14^2
		169	144	156	1^{169}	-13^{26}	OA(14,13)?
?	196	39	2	9	3^{147}	-10^{48}	
		156	125	120	9^{48}	-4^{147}	
+	196	39	14	6	11^{39}	-3^{156}	OA(14,3)
		156	122	132	2^{156}	-12^{39}	OA(14,12)?
?	196	45	4	12	3^{150}	-11^{45}	
		150	116	110	10^{45}	-4^{150}	
+	196	52	18	12	10^{52}	-4^{143}	OA(14,4)
		143	102	110	3^{143}	-11^{52}	OA(14,11)?
?	196	60	14	20	4^{135}	-10^{60}	
		135	94	90	9^{60}	-5^{135}	
+	196	60	23	16	11^{48}	-4^{147}	S(2,4,49); Huffman-Tonchev [1150]: intersection 3 graph of a 2-(49,9,6) design with block intersections 1, 3
		135	90	99	3^{147}	-12^{48}	
+	196	65	24	20	9^{65}	-5^{130}	OA(14,5)
		130	84	90	4^{130}	-10^{65}	OA(14,10)?
?	196	75	26	30	5^{120}	-9^{75}	
		120	74	72	8^{75}	-6^{120}	
?	196	78	32	30	8^{78}	-6^{117}	OA(14,6)?
		117	68	72	5^{117}	-9^{78}	OA(14,9)?
?	196	81	42	27	18^{24}	-3^{171}	
		114	59	76	2^{171}	-19^{24}	
$-$	196	85	18	51	1^{187}	-34^{8}	Krein$_2$; Abs
		110	75	44	33^{8}	-2^{187}	Krein$_1$; Abs
?	196	90	40	42	6^{105}	-8^{90}	$RSHCD-$?; 2-graph?
		105	56	56	7^{90}	-7^{105}	2-graph?
+	196	91	42	42	7^{91}	-7^{104}	OA(14,7)?; $RSHCD+$; 2-graph
		104	54	56	6^{104}	-8^{91}	OA(14,8)?; 2-graph
+	197	98	48	49	6.518^{98}	-7.518^{98}	Paley(197); 2-graph$-*$
$-$	201	100	49	50	6.589^{100}	-7.589^{100}	Conf
?	204	28	2	4	4^{119}	-6^{84}	
		175	150	150	5^{84}	-5^{119}	
?	204	63	22	18	9^{68}	-5^{135}	
		140	94	100	4^{135}	-10^{68}	S(2,10,136)?
?	205	68	15	26	3^{164}	-14^{40}	
		136	93	84	13^{40}	-4^{164}	
?	205	96	50	40	14^{40}	-4^{164}	
		108	51	63	3^{164}	-15^{40}	

∃	v	k	λ	μ	r^f	s^g	Comments
?	205	102	50	51	6.659^{102}	-7.659^{102}	2-graph$-*$?
?	208	45	8	10	5^{117}	-7^{90}	
		162	126	126	6^{90}	-6^{117}	
+	208	75	30	25	10^{64}	-5^{143}	S(2,5,65); $NU(3,4)$
		132	81	88	4^{143}	-11^{64}	
?	208	81	24	36	3^{168}	-15^{39}	
		126	80	70	14^{39}	-4^{168}	
−	209	16	3	1	5^{76}	-3^{132}	$\mu = 1$
		192	176	180	2^{132}	-6^{76}	
?	209	52	15	12	8^{76}	-5^{132}	
		156	115	120	4^{132}	-9^{76}	
+	209	100	45	50	5^{132}	-10^{76}	2-graph$-*$
		108	57	54	9^{76}	-6^{132}	2-graph$-*$
−	209	104	51	52	6.728^{104}	-7.728^{104}	Conf
?	210	33	0	6	3^{154}	-9^{55}	
		176	148	144	8^{55}	-4^{154}	
!	210	38	19	4	17^{20}	-2^{189}	$\binom{21}{2}$
		171	136	153	1^{189}	-18^{20}	
?	210	76	26	28	6^{114}	-8^{95}	
		133	84	84	7^{95}	-7^{114}	
?	210	77	28	28	7^{99}	-7^{110}	
		132	82	84	6^{110}	-8^{99}	
?	210	95	40	45	5^{133}	-10^{76}	2-graph?
		114	63	60	9^{76}	-6^{133}	2-graph?
+	210	99	48	45	9^{77}	-6^{132}	Sym(7); Klin; 2-graph
		110	55	60	5^{132}	-10^{77}	2-graph
−	213	106	52	53	6.797^{106}	-7.797^{106}	Conf
?	216	40	4	8	4^{140}	-8^{75}	
		175	142	140	7^{75}	-5^{140}	
?	216	43	10	8	7^{86}	-5^{129}	
		172	136	140	4^{129}	-8^{86}	
−	216	70	40	14	28^{12}	-2^{203}	Abs
		145	88	116	1^{203}	-29^{12}	Abs
?	216	75	18	30	3^{175}	-15^{40}	
		140	94	84	14^{40}	-4^{175}	
?	216	86	40	30	14^{43}	-4^{172}	
		129	72	84	3^{172}	-15^{43}	
?	216	90	39	36	9^{80}	-6^{135}	S(2,6,81)?
		125	70	75	5^{135}	-10^{80}	
?	217	66	15	22	4^{154}	-11^{62}	
		150	105	100	10^{62}	-5^{154}	
?	217	88	39	33	11^{62}	-5^{154}	
		128	72	80	4^{154}	-12^{62}	
−	217	108	53	54	6.865^{108}	-7.865^{108}	Conf
?	220	72	22	24	6^{120}	-8^{99}	
		147	98	98	7^{99}	-7^{120}	
+	220	84	38	28	14^{44}	-4^{175}	Tonchev: intersection 3 graph of a 2-(45,9,8) design with block intersections 1, 3
		135	78	90	3^{175}	-15^{44}	
+	221	64	24	16	12^{51}	-4^{169}	S(2,4,52)
		156	107	117	3^{169}	-13^{51}	
?	221	110	54	55	6.933^{110}	-7.933^{110}	2-graph$-*$?
+	222	51	20	9	14^{36}	-3^{185}	S(2,3,37)
		170	127	140	2^{185}	-15^{36}	
!	225	28	13	2	13^{28}	-2^{196}	15^2

∃	v	k	λ	μ	r^f	s^g	Comments
		196	169	182	1^{196}	-14^{28}	OA(15,14)?
+	225	42	15	6	12^{42}	-3^{182}	OA(15,3)
		182	145	156	2^{182}	-13^{42}	OA(15,13)?
?	225	48	3	12	3^{176}	-12^{48}	
		176	139	132	11^{48}	-4^{176}	
−	225	56	1	18	2^{200}	-19^{24}	Krein_2
		168	129	114	18^{24}	-3^{200}	Krein_1
+	225	56	19	12	11^{56}	-4^{168}	OA(15,4)
		168	123	132	3^{168}	-12^{56}	OA(15,12)?
?	225	64	13	20	4^{160}	-11^{64}	
		160	115	110	10^{64}	-5^{160}	
+	225	70	25	20	10^{70}	-5^{154}	OA(15,5)
		154	103	110	4^{154}	-11^{70}	OA(15,11)?
?	225	80	25	30	5^{144}	-10^{80}	
		144	93	90	9^{80}	-6^{144}	
+	225	84	33	30	9^{84}	-6^{140}	OA(15,6)
		140	85	90	5^{140}	-10^{84}	OA(15,10)?
−	225	96	19	57	1^{216}	-39^{8}	Krein_2; Abs
		128	88	52	38^{8}	-2^{216}	Krein_1; Abs
?	225	96	39	42	6^{128}	-9^{96}	
		128	73	72	8^{96}	-7^{128}	
?	225	96	51	33	21^{24}	-3^{200}	
		128	64	84	2^{200}	-22^{24}	
+	225	98	43	42	8^{98}	-7^{126}	OA(15,7)?; Pasechnik(15)
		126	69	72	6^{126}	-9^{98}	OA(15,9)?
+	225	112	55	56	7^{112}	-8^{112}	skewhad^2; OA(15,8)?; 2-graph$-*$
+	226	105	48	49	7^{113}	-8^{112}	switch $\text{skewhad}^2 + *$; 2-graph
		120	63	64	7^{112}	-8^{113}	S(2,8,113)?; 2-graph
+	229	114	56	57	7.066^{114}	-8.066^{114}	Paley(229); 2-graph$-*$
+	231	30	9	3	9^{55}	-3^{175}	M_{22}; Cameron
		200	172	180	2^{175}	-10^{55}	
!	231	40	20	4	18^{21}	-2^{209}	$\binom{22}{2}$
		190	153	171	1^{209}	-19^{21}	
?	231	70	21	21	7^{110}	-7^{120}	
		160	110	112	6^{120}	-8^{110}	
?	231	90	33	36	6^{132}	-9^{98}	
		140	85	84	8^{98}	-7^{132}	
?	232	33	2	5	4^{144}	-7^{87}	
		198	169	168	6^{87}	-5^{144}	
?	232	63	14	18	5^{144}	-9^{87}	
		168	122	120	8^{87}	-6^{144}	
?	232	77	36	20	19^{28}	-3^{203}	
		154	96	114	2^{203}	-20^{28}	
?	232	81	30	27	9^{87}	-6^{144}	
		150	95	100	5^{144}	-10^{87}	S(2,10,145)?
+	233	116	57	58	7.132^{116}	-8.132^{116}	Paley(233); 2-graph$-*$
?	235	42	9	7	7^{94}	-5^{140}	
		192	156	160	4^{140}	-8^{94}	
?	235	52	9	12	5^{140}	-8^{94}	
		182	141	140	7^{94}	-6^{140}	
?	236	55	18	11	11^{59}	-4^{176}	
		180	135	144	3^{176}	-12^{59}	S(2,12,177)?
−	237	118	58	59	7.197^{118}	-8.197^{118}	Conf
?	238	75	20	25	5^{153}	-10^{84}	
		162	111	108	9^{84}	-6^{153}	

$\exists$	v	k	λ	μ	r^f	s^g	Comments
+	241	120	59	60	7.262^{120}	-8.262^{120}	Paley(241); 2-graph$-*$
+	243	22	1	2	4^{132}	-5^{110}	$3^5.2.M_{11}$ (rk 3); Berlekamp-vanLint-Seidel [220]; Golay code: projective ternary [11,5] code with weights 6, 9
		220	199	200	4^{110}	-5^{132}	
?	243	66	9	21	3^{198}	-15^{44}	
		176	130	120	14^{44}	-4^{198}	
−	243	88	52	20	34^{11}	-2^{231}	Abs
		154	85	119	1^{231}	-35^{11}	Abs
+	243	110	37	60	2^{220}	-25^{22}	$3^5.2.M_{11}$ (rk 3); Delsarte; projective ternary [55,5] code with weights 36, 45
		132	81	60	24^{22}	-3^{220}	
?	243	112	46	56	4^{182}	-14^{60}	2-graph$-*$?
		130	73	65	13^{60}	-5^{182}	2-graph$-*$?
?	244	108	42	52	4^{183}	-14^{60}	2-graph?
		135	78	70	13^{60}	-5^{183}	2-graph?
?	244	117	60	52	13^{61}	-5^{182}	2-graph?
		126	60	70	4^{182}	-14^{61}	2-graph?
?	245	52	3	13	3^{195}	-13^{49}	
		192	152	144	12^{49}	-4^{195}	
?	245	64	18	16	8^{100}	-6^{144}	
		180	131	135	5^{144}	-9^{100}	
?	245	108	39	54	3^{204}	-18^{40}	2-graph$-*$?
		136	81	68	17^{40}	-4^{204}	2-graph$-*$?
?	245	122	60	61	7.326^{122}	-8.326^{122}	2-graph$-*$?
?	246	85	20	34	3^{204}	-17^{41}	
		160	108	96	16^{41}	-4^{204}	
?	246	105	36	51	3^{205}	-18^{40}	2-graph?
		140	85	72	17^{40}	-4^{205}	2-graph?
?	246	119	64	51	17^{41}	-4^{204}	2-graph?
		126	57	72	3^{204}	-18^{41}	2-graph?
+	247	54	21	9	15^{38}	-3^{208}	S(2,3,39)
		192	146	160	2^{208}	-16^{38}	
?	249	88	27	33	5^{165}	-11^{83}	
		160	104	100	10^{83}	-6^{165}	
−	249	124	61	62	7.390^{124}	-8.390^{124}	Conf
?	250	81	24	27	6^{144}	-9^{105}	
		168	113	112	8^{105}	-7^{144}	
?	250	96	44	32	16^{45}	-4^{204}	
		153	88	102	3^{204}	-17^{45}	
!	253	42	21	4	19^{22}	-2^{230}	$\binom{23}{2}$
		210	171	190	1^{230}	-20^{22}	
−	253	90	17	40	2^{230}	-25^{22}	Krein_2
		162	111	90	24^{22}	-3^{230}	Krein_1
+	253	112	36	60	2^{230}	-26^{22}	$S(4,7,23)$; M_{23}
		140	87	65	25^{22}	-3^{230}	Witt 4-(23,7,1): intersection 3 graph of a 2-(23,7,21) design with block intersections 1, 3
−	253	126	62	63	7.453^{126}	-8.453^{126}	Conf
+	255	126	61	63	7^{135}	-9^{119}	$O_9(2)$ polar graph; $Sp_8(2)$ polar graph; 2-graph$-*$
		128	64	64	8^{119}	-8^{135}	S(2,8,120); 2-graph$-*$
!	256	30	14	2	14^{30}	-2^{225}	16^2; from a partial spread: projective 4-ary [10,4] code with weights 4, 8; from a partial spread of 4-spaces: projective binary [30,8] code with weights 8, 16
		225	196	210	1^{225}	-15^{30}	OA(16,15)
+	256	45	16	6	13^{45}	-3^{210}	OA(16,3); $\text{Bilin}_{2\times 4}(2)$; Brouwer [349]; from a partial spread: projective 4-ary [15,4] code with weights 8, 12; from a partial spread of 4-spaces: projective binary [45,8] code with weights 16, 24

$\exists$	v	k	λ	μ	r^f	s^g	Comments
		210	170	182	2^{210}	-14^{45}	OA(16,14)
+	256	51	2	12	3^{204}	-13^{51}	vanLint-Schrijver(1); $VO_4^-(4)$ affine polar graph; projective 4-ary [17,4] code with weights 12, 16
		204	164	156	12^{51}	-4^{204}	vanLint-Schrijver(4)
+	256	60	20	12	12^{60}	-4^{195}	OA(16,4); from a partial spread: projective 4-ary [20,4] code with weights 12, 16; Brouwer [349]; from a partial spread of 4-spaces: projective binary [60,8] code with weights 24, 32
		195	146	156	3^{195}	-13^{60}	OA(16,13)
−	256	66	2	22	2^{231}	-22^{24}	Krein_2
		189	144	126	21^{24}	-3^{231}	Krein_1
+	256	68	12	20	4^{187}	-12^{68}	Brouwer [349]; projective binary [68,8] code with weights 32, 40
		187	138	132	11^{68}	-5^{187}	
+	256	75	26	20	11^{75}	-5^{180}	OA(16,5); $\text{Bilin}_{2\times 2}(4)$; $VO_4^+(4)$ affine polar graph; from a partial spread: projective 4-ary [25,4] code with weights 16, 20; from a partial spread of 4-spaces: projective binary [75,8] code with weights 32, 40
		180	124	132	4^{180}	-12^{75}	OA(16,12)
+	256	85	24	30	5^{170}	-11^{85}	vanLint-Schrijver(1); possibly projective binary [85,8] code with weights 40, 48
		170	114	110	10^{85}	-6^{170}	vanLint-Schrijver(2)
+	256	90	34	30	10^{90}	-6^{165}	OA(16,6); from a partial spread: projective 4-ary [30,4] code with weights 20, 24; from a partial spread of 4-spaces: projective binary [90,8] code with weights 40, 48
		165	104	110	5^{165}	-11^{90}	OA(16,11)
+	256	102	38	42	6^{153}	-10^{102}	$2^8.L_2(17)$ (rk 3); Liebeck; vanLint-Schrijver(2); possibly projective 4-ary [34,4] code with weights 24, 28
		153	92	90	9^{102}	-7^{153}	vanLint-Schrijver(3)
+	256	105	44	42	9^{105}	-7^{150}	OA(16,7); from a partial spread: projective 4-ary [35,4] code with weights 24, 28; Brouwer [349]; from a partial spread of 4-spaces: projective binary [105,8] code with weights 48, 56
		150	86	90	6^{150}	-10^{105}	OA(16,10)
+	256	119	54	56	7^{136}	-9^{119}	$VO_8^-(2)$ affine polar graph; projective binary [119,8] code with weights 56, 64; $RSHCD-$; 2-graph
		136	72	72	8^{119}	-8^{136}	2-graph
+	256	120	56	56	8^{120}	-8^{135}	OA(16,8); from a partial spread: projective 4-ary [40,4] code with weights 28, 32; from a partial spread of 4-spaces: projective binary [120,8] code with weights 56, 64; $RSHCD+$; 2-graph
		135	70	72	7^{135}	-9^{120}	OA(16,9); Goethals-Seidel(2,15); $VO_8^+(2)$ affine polar graph; 2-graph
+	257	128	63	64	7.516^{128}	-8.516^{128}	Paley(257); 2-graph$-*$
?	259	42	5	7	5^{147}	-7^{111}	
		216	180	180	6^{111}	-6^{147}	
?	260	70	15	20	5^{168}	-10^{91}	
		189	138	135	9^{91}	-6^{168}	
?	261	52	11	10	7^{116}	-6^{144}	
		208	165	168	5^{144}	-8^{116}	
?	261	64	14	16	6^{144}	-8^{116}	
		196	147	147	7^{116}	-7^{144}	
?	261	80	25	24	8^{116}	-7^{144}	
		180	123	126	6^{144}	-9^{116}	
?	261	84	39	21	21^{29}	-3^{231}	

∃	v	k	λ	μ	r^f	s^g	Comments
		176	112	132	2^{231}	-22^{29}	
?	261	130	64	65	7.578^{130}	-8.578^{130}	2-graph$-*$?
?	265	96	32	36	6^{159}	-10^{105}	
		168	107	105	9^{105}	-7^{159}	
?	265	132	65	66	7.639^{132}	-8.639^{132}	2-graph$-*$?
?	266	45	0	9	3^{209}	-12^{56}	
		220	183	176	11^{56}	-4^{209}	
+	269	134	66	67	7.701^{134}	-8.701^{134}	Paley(269); 2-graph$-*$
?	273	72	21	18	9^{104}	-6^{168}	
		200	145	150	5^{168}	-10^{104}	
?	273	80	19	25	5^{182}	-11^{90}	
		192	136	132	10^{90}	-6^{182}	
+	273	102	41	36	11^{90}	-6^{182}	S(2,6,91)
		170	103	110	5^{182}	-12^{90}	
?	273	136	65	70	6^{168}	-11^{104}	
		136	69	66	10^{104}	-7^{168}	
−	273	136	67	68	7.761^{136}	-8.761^{136}	Conf
!	275	112	30	56	2^{252}	-28^{22}	$q_{22}^2 = 0$; 2-graph$-*$
		162	105	81	27^{22}	-3^{252}	$McL.2/U_4(3).2$; McLaughlin graph; unique by Goethals & Seidel [917]; $q_{11}^1 = 0$; 2-graph$-*$
!	276	44	22	4	20^{23}	-2^{252}	$\binom{24}{2}$
		231	190	210	1^{252}	-21^{23}	
?	276	75	10	24	3^{230}	-17^{45}	
		200	148	136	16^{45}	-4^{230}	
?	276	75	18	21	6^{160}	-9^{115}	
		200	145	144	8^{115}	-7^{160}	
−	276	110	28	54	2^{253}	-28^{22}	Krein_2; Abs
		165	108	84	27^{22}	-3^{253}	Krein_1; Abs
?	276	110	52	38	18^{45}	-4^{230}	
		165	92	108	3^{230}	-19^{45}	
+	276	135	78	54	27^{23}	-3^{252}	Conway-G&S; 2-graph
		140	58	84	2^{252}	-28^{23}	2-graph
+	277	138	68	69	7.822^{138}	-8.822^{138}	Paley(277); 2-graph$-*$
+	279	128	52	64	4^{216}	-16^{62}	2-graph$-*$
		150	85	75	15^{62}	-5^{216}	2-graph$-*$
+	280	36	8	4	8^{90}	-4^{189}	$J_2/3.PGL_2(9)$ (rk 4); $U_4(3)$ polar graph
		243	210	216	3^{189}	-9^{90}	
?	280	62	12	14	6^{155}	-8^{124}	
		217	168	168	7^{124}	-7^{155}	
?	280	63	14	14	7^{135}	-7^{144}	
		216	166	168	6^{144}	-8^{135}	
+	280	117	44	52	5^{195}	-13^{84}	
		162	96	90	12^{84}	-6^{195}	Sym(9) (rk 5); Mathon & Rosa
?	280	124	48	60	4^{217}	-16^{62}	2-graph?
		155	90	80	15^{62}	-5^{217}	2-graph?
+	280	135	70	60	15^{63}	-5^{216}	$J_2/3.PGL_2(9)$ (rk 4); 2-graph
		144	68	80	4^{216}	-16^{63}	2-graph

10.3 See Also

§II.5	Steiner systems.
§III.3	Mutually orthogonal latin squares.
§V.1	Hadamard matrices.
§V.6	Conference matrices.
§VI.1	Association schemes.
§VI.23	Generalized quadrangles.
§VI.48	Quasi-symmetric designs.
§VII.12	Two-graphs.
[354, 1144]	Early surveys of the construction of strongly regular graphs.
[350]	Parameters of infinite series, the isomorphisms, existence and nonexistence results.

References Cited: [220, 345, 347, 349, 350, 351, 352, 354, 407, 415, 423, 452, 519, 522, 604, 605, 610, 670, 671, 674, 890, 891, 914, 916, 917, 999, 1000, 1016, 1092, 1094, 1095, 1112, 1144, 1150, 1212, 1318, 1356, 1525, 1565, 1658, 1660, 1706, 1707, 1711, 1718, 1844, 1855, 1857, 1887, 1904, 1918, 1934, 1986, 1987, 2065, 2116]

11 Directed strongly regular graphs

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11.1 Definitions and Basics

11.1 A directed strongly regular graph (dsrg) is a (0,1)-matrix A with 0's on the diagonal such that the linear span of I, A and J is closed under matrix multiplication.

11.2 The (integral) parameters v, k, t, λ, μ are defined by: A is a matrix of order v, and $AJ = JA = kJ$, and $A^2 = tI + \lambda A + \mu(J - I - A)$. The object is a dsrg($v$,$k$,$t$,$\lambda$,$\mu$).

11.3 **Remark** Warning: many authors use the parameter ordering (v, k, μ, λ, t).

11.4 The *complement* of a dsrg A is $J - I - A$. The complement is also a dsrg.

11.5 **Proposition** If A has parameters v, k, t, λ, μ then its complement $J - I - A$ has parameters v, $v-k-1$, $v-2k-1+t$, $v-2k-2+\mu$, $v-2k+\lambda$. In the tables below, complementary pairs are given together.

11.6 The *reverse* of a dsrg A is A^T. The reverse is also a dsrg.

11.7 **Remark** The concept of *directed* strongly regular graphs is due to Duval [761]. Hammersley [1022] studied the "love problem", a problem related to this present topic. This paper contains pictures of dsrg(8,3,2,1,1) and dsrg(15,4,2,1,1).

11.8 **Remark** Clearly, $0 \le t \le k$. In what follows it is assumed that $0 < t < k$. The case $t = k$ is that of strongly regular graphs, where A is symmetric (see §VII.10). The case $t = 0$ is that of Hadamard matrices H with 1's on the diagonal and skew symmetric off-diagonal. (See §V.1 and [1109].)

▷ **Spectrum**

11.9 Remark The algebra spanned by I, A and J is 3-dimensional. The matrix A is diagonalizable, and has precisely 3 distinct eigenvalues, say k, r, s, with multiplicities 1, f and g, respectively, with $f \neq g$. The eigenvalues r, s different from k are roots of $x^2 + (\mu - \lambda)x + \mu - t = 0$. They are integers, and r is chosen nonnegative, while s is chosen negative. (One sees that $\mu \leq t$.)

▷ **The 2-Dimensional and 1-Dimensional Cases**

11.10 Remark An important subclass is when the linear span of A and J is closed under matrix multiplication. This happens when $t = \mu$, and then $A^2 = (\lambda - \mu)A + \mu J$ so that the eigenvalues of A are k, $\lambda - \mu$, and 0. Conversely, when A has eigenvalue 0, we are in this case. Put $d = \mu - \lambda$. Then the multiplicities of the eigenvalues k, $-d$, 0 are 1, k/d and $v - 1 - k/d$ respectively, and A has rank $1 + k/d$.

11.11 Remark If A is a 0-1 matrix, then $B = J - 2A$ is a ± 1 matrix, and it is possible that the 1-space generated by B is closed under matrix multiplication. This happens when $t = \mu$ and $v - 4k + 4t = d$ (with d as above - clearly this is a subcase of the 2-dimensional case). Conversely, if B is a ± 1 matrix with constant row sums and 1's on the diagonal such that B^2 is a multiple of B, then $A = (J - B)/2$ is the adjacency matrix of a dsrg in this subcase. The set of such ± 1 matrices B is closed under taking tensor products.

▷ **Combinatorial Parameter Conditions**

11.12 Remark Definition 11.2 implies that $0 < \mu \leq t$. (If $\mu = 0$ then $A = J - I$, which was doublecheck
excluded.) It also follows that $0 \leq \lambda < t < k < v$. (Indeed, $\lambda < t$, since $\lambda + 1 - t =$ definition number
$(r+1)(s+1) \leq 0$.) The matrix equation immediately gives $k(k - \lambda + \mu) = \mu(v-1) + t$. Duval [761] gave one more condition, and parameter sets violating it are not mentioned in Table 11.47. The next theorem follows from counting the number of paths of length 2 between two points

11.13 Theorem If A is a dsrg(v,k,t,λ,μ), then $-2(k - t - 1) \leq \mu - \lambda \leq 2(k - t)$.

▷ **Not a Cayley Graph of an Abelian Group**

11.14 Theorem (Klin et al. [1306]) A directed strongly regular graph cannot be a Cayley graph of an abelian group.

11.15 Remark Theorem 11.14 is proved by observing that if A_s is the symmetric part of A (with row sums t), then A and A_s have real eigenvalues, while $A - A_s$ has a non-real eigenvalue (since $(A - A_s)^2$ has zero trace and the positive eigenvalue $(k-t)^2$), so that A and A_s do not commute.

11.2 Constructions

11.16 Construction [C1] (Duval [761]) For every $k > 2$, there exists a dsrg with $v = k(k+1)$, $k = k$, $t = 1$, $\mu = 1$, $\lambda = 0$.

11.17 Remark Construction C1 is a special case of C6(i) obtained by taking a 2-$(k+1, 2, 1)$ design.

11.18 Construction [C2] (Duval [761]) Let q be a prime power, $q \equiv 1 \pmod 4$. Then there exists a dsrg with $v = 2q$, $k = q - 1$, $t = \mu = \lambda + 1 = k/2$.

11.19 Construction [C3] (Jørgensen [1210]) Let μ, k be positive integers such that $\mu|(k-1)$. Then there exists a dsrg with $v = (k+1)(k-1)/\mu$, $t = \mu + 1$, $\lambda = \mu$. Jørgensen's construction is as follows: take as vertices the integers mod v and let $x \to y$ be an edge when $x + ky = 1, 2, ..., k \pmod v$.

11.20 Construction [C4] (Klin et al. [1306]) Let n be an odd integer. Then there exists a dsrg with $v = 2n$, $k = n - 1$, $t = \mu = \lambda + 1 = k/2$.

11.21 Remark The graphs from Construction C4, and those of [1110] with parameters $(4t, 2t-1, t, t-1, t-1)$ (also covered by [C3] above), are Cayley graphs of nonabelian groups.

11.22 Construction [C5] (Klin et al. [1307]) If there exists a generalized quadrangle GQ(a,b), then there exists a dsrg with $v = (a+1)(b+1)(ab+1)$, $k = ab+a+b$, $t = \lambda+1 = a+b$, $\mu = 1$. This graph is obtained by looking at (point,line) flags, where $(p, L) \to (q, M)$ when the flags are distinct and q is on L.

11.23 Construction [C6] (Fiedler et al. [810]) If there exists a 2-(V, B, R, K, Λ) design, then (i) there exists a dsrg with $v = VR$, $k = R(K-1)$, $t = \mu = \Lambda(K-1)$, $\lambda = \Lambda(K-2)$ and also (ii) a dsrg with $v = VR$, $k = RK - 1$, $t = \lambda+1 = \Lambda(K-1) + R - 1$, $\mu = \Lambda K$. The former is obtained by looking at (point,block) flags, where $(p, B) \to (q, C)$ when p and q are distinct and q is on B. The latter is obtained by looking at (point,block) flags, where $(p, B) \to (q, C)$ when the flags are distinct and q is on B.

11.24 Construction [C7] (Fiedler et al. [810]) If there exists a partial geometry with parameters pg(K,R,T) then there exists a dsrg with $v = RK(1 + (K-1)(R-1)/T)$, $k = RK - 1$, $t = \lambda + 1 = K + R - 2$, $\mu = T$. This graph is obtained by looking at (point,line) flags, where $(p, L) \to (q, M)$ when the flags are distinct and q is on L. Of course C5 is the special case $T = 1$ of this.

11.25 Construction [C8] (Duval [761]) If there exists a dsrg with parameters v, k, t, λ, μ, and $t = \mu$, then for each positive integer m there also is a dsrg with parameters mv, mk, mt, $m\lambda$, $m\mu$. Construction: given A, consider $A \otimes J$, with J of order m.

11.26 Construction [C9] (Godsil et al. [910]) Suppose n, d, and c are positive integers such that $c < n-1$ and $c|d(n-1)$. Then there exists a dsrg with parameters $v = dn(n-1)/c$, $k = d(n-1)$, $t = \mu = cd$, $\lambda = (c-1)d$.

11.27 Construction [C10] (Fiedler et al. [809]) Let q be an odd prime power. Then there exists a dsrg with $v = (q-1)(q^2-1)$, $k = q^2 - 2$, $t = \lambda + 1 = 2q - 2$, $\mu = q$.

11.28 Construction [C11] (Fiedler et al. [809]) Let q be an odd prime power. There there exists a dsrg with $v = 2q^2$, $k = 3q - 2$, $t = 2q - 1$, $\lambda = q - 1$, $\mu = 3$.

11.29 Construction [C12] If there exists a srg with parameters v, k, λ, μ, and $\mu = \lambda+1$, then there is a dsrg with parameters vk, $(v-k-1)k$, $(v-k-1)(k-\mu)$, $(v-k-2)(k-\mu)$, $(v-k-1)(k-\mu)$. Construction: take the edges of the srg, and let $xy \to uv$ when u is at distance 2 from y.

11.30 Example Using Construction C12, the Petersen graph produces a dsrg(30,18,12,10,12).

11.31 Construction [C13] Let r be an odd prime power. Then there exists a dsrg with parameters $2r^2$, $(r-1)(2r+1)/2$, $(r-1)(3r+1)/4$, $r(r-1)/2 - 1$, $r(r-1)/2$.

11.32 Remark Construction C13 is obtained as follows: Take two copies of the finite field of order r^2. Join vertices in one copy when they differ by a square, in the other when they differ by a non-square. Thus, both halves are undirected (Paley) graphs. View the field as AG(2,r) and pick two sets, B and C, each the union of $(r-1)/2$ mutually parallel lines, where B and C use different directions. Make an arrow from x in the first copy to y' in the second copy when $y-x$ is a member of B, and an arrow from y' to x when $y-x$ is a member of C.

11.33 Construction [C14] The tensor product of two ± 1 matrices belonging to dsrgs in the 1-dimensional case, is again a dsrg in the 1-dimensional case. Here one may also take I of order 2.

11.34 Construction [C15] (Godsil et al. [910]) Given a dsrg with adjacency matrix A, where $\mu = \lambda$ and $v = 4(k-\mu)$, and a Hadamard matrix H of order c with constant row and column sums d, and ones on the diagonal, put $B = J - 2A$ and $B' = B \otimes H$. The parameter conditions say that B is a ± 1 matrix with 1's on the diagonal, constant row and column sums, and with B^2 a multiple of I. Clearly, the same holds for B'. Hence A', given by $B' = J - 2A'$, is again the adjacency matrix of a dsrg. The parameters are $v' = vc$, $k' = 2c(k-\mu) + d(2\mu - k)$, $t' = c(k+t-2\mu) + d(2\mu - k)$, $\lambda' = \mu' = c(k-\mu) + d(2\mu - k)$. Note that $c = d^2$.

11.3 Nonexistence Conditions

11.35 Theorem [N1] (Jørgensen [1211]) If $g = 2$ then $(v,k,t,\lambda,\mu) = (6m, 2m, m, 0, m)$ or $(v,k,t,\lambda,\mu) = (8m, 4m, 3m, m, 3m)$.

11.36 Theorem [N2] (Jørgensen [1211]) If $g = 3$ then $(v,k,t,\lambda,\mu) = (6m, 3m, 2m, m, 2m)$ or $(v,k,t,\lambda,\mu) = (12m, 3m, m, 0, m)$.

11.37 Theorem [N3](Jørgensen [1211]) $(k-t)(\mu - k + t) \le \lambda t$.

11.38 Remark Theorem 11.37 is proven as follows: Fix a point x and look at triangles xyz where xy is a directed edge and xz is an undirected edge, and yz may be directed (in the yz direction) or undirected. There are at least the LHS and at most the RHS such triangles.

11.39 Theorem [N4] (Jørgensen [1211]) If $\lambda = 0$ and $\mu = k - t$, then v is a multiple of 3μ.

11.40 Remark Theorem 11.39 is proven as follows: since $\lambda = 0$, the μ triangles on a directed edge contain directed edges only, and one sees that the directed edges form a graph where each connected component has 3μ vertices since it is the μ-coclique extension of a directed triangle.

11.41 Theorem (Absolute Bound, Jørgensen [1211])
(i) If $s \ne -1$, then $v \le f(f+3)/2$. If $v(r+s^2) \ne 2(r-s)(k-s)$ also, then $v \le f(f+1)/2$.
(ii) If $r \ne 0$, then $v \le g(g+3)/2$. If $v(s+r^2) \ne 2(s-r)(k-r)$ also, then $v \le g(g+1)/2$.

11.4 Existence, Uniqueness and Enumeration

11.42 Examples The two smallest dsrg's (with parameters (6,2,1,0,1) and (8,4,3,1,3)) are unique. Matrices are for these dsrgs A and B, respectively, are given below. The matrix C is a dsrg with parameters (10,4,2,1,2) where points come in pairs with equal in-neighbours, but there is no such pairing for out-neighbours. There are 15 other examples with the same parameters. In a few cases uniqueness is known.

$$A = \begin{array}{|c|}\hline 001100\\ 000011\\ 000011\\ 110000\\ 110000\\ 001100\\ \hline\end{array} \qquad B = \begin{array}{|c|}\hline 00001111\\ 00001111\\ 01010101\\ 10101010\\ 01010101\\ 10101010\\ 11110000\\ 11110000\\ \hline\end{array} \qquad C = \begin{array}{|c|}\hline 0000001111\\ 0011110000\\ 1100110000\\ 0000001111\\ 0000001111\\ 1111000000\\ 1100000011\\ 0011110000\\ 0011110000\\ 1100001100\\ \hline\end{array}.$$

11.43 Theorem (Gimbert [904]) The dsrg($k(k+1)$,k,1,0,1) is unique for all positive integers k.

11.44 Theorem (Jørgensen [1211]) The dsrg($m(m+1)t$,mt,t,0,t) is unique for all positive integers m and t.

11.45 Theorem [1209] There are precisely 1292 nonisomorphic dsrg(15,5,2,1,2), namely 1174, 100, 10, 5, 3 with full automorphism group of order 1, 2, 3, 4, 5, respectively.

11.46 Table Table of (mostly Jørgensen's) enumeration results (see also [1208]):

v	k	t	λ	μ	#	comment
6	2	1	0	1	1	
8	3	2	1	1	1	Hammersley [1022]
10	4	2	1	2	16	Jørgensen [1209]
12	3	1	0	1	1	
12	4	2	0	2	1	
12	5	3	2	2	20	Jørgensen [1209]
14	6	3	2	3	16495	Jørgensen [1209]
15	4	2	1	1	5	Jørgensen [1209]
15	5	2	1	2	1292	Jørgensen [1209]
16	7	5	4	2	1	Jørgensen [1209]
18	4	3	0	1	1	Jørgensen [1209]
18	5	3	2	1	2	Jørgensen [1209]
18	6	3	0	3	1	
20	4	1	0	1	1	
24	6	2	0	2	1	

11.47 Table Directed strongly regular graphs with $v \leq 32$. On-line tables at [1109].

$\exists$	v	k	λ	μ	r^f	s^g	Comments
6	2	1	0	1	0^3	-1^2	C1, C4, C6(i) for 2-(3,2,1), C9
	3	2	1	2	0^2	-1^3	C6(ii) for 2-(3,2,1), C7 for pg(2,2,2)
8	3	2	1	1	1^2	-1^5	C3, C5 for GQ(1,1)
	4	3	1	3	0^5	-2^2	
10	4	2	1	2	0^5	-1^4	C2, C4, C9, C12 from srg(5,2,0,1)
	5	3	2	3	0^4	-1^5	
12	3	1	0	1	0^8	-1^3	C1, C6(i) for 2-(4,2,1), C9
	8	6	5	6	0^3	-1^8	C6(ii) for 2-(4,3,2)
12	4	2	0	2	0^9	-2^2	C6(i) for 2-(3,2,2), C8 for m=2, C9
	7	5	4	4	1^2	-1^9	C6(ii) for 2-(3,2,2)
12	5	3	2	2	1^3	-1^8	C3, C6(ii) for 2-(4,2,1), C7 for pg(3,2,2)
	6	4	2	4	0^8	-2^3	C6(i) for 2-(4,3,2), C8 for m=2, C9
14	5	4	1	2	1^7	-2^6	does not exist (Klin et al. [1306])
	8	7	4	5	1^6	-2^7	
14	6	3	2	3	0^7	-1^6	C4, C9
	7	4	3	4	0^6	-1^7	

v	k	t	λ	μ	r^f	s^g	comments
15	4	2	1	1	1^5	-1^9	C3
	10	8	6	8	0^9	-2^5	
15	5	2	1	2	0^9	-1^5	Jørgensen [1209]
	9	6	5	6	0^5	-1^9	
16	6	3	1	3	0^{12}	-2^3	does not exist (Fiedler et al. [809], or N2)
	9	6	5	5	1^3	-1^{12}	
16	7	4	3	3	1^4	-1^{11}	C3, C10
	8	5	3	5	0^{11}	-2^4	
16	7	5	4	2	3^2	-1^{13}	
	8	6	2	6	0^{13}	-4^2	C8 for m=2
18	4	3	0	1	1^{10}	-2^7	Duval [761], Theorem 4.4.
	13	12	9	10	1^7	-2^{10}	
18	5	3	2	1	2^4	-1^{13}	C5 for GQ(1,2)
	12	10	7	10	0^{13}	-3^4	
18	6	3	0	3	0^{15}	-3^2	C6(i) for 2-(3,2,3), C8 for m=3, C9
	11	8	7	6	2^2	-1^{15}	C6(ii) for 2-(3,2,3)
18	7	5	2	3	1^9	-2^8	C11, C13
	10	8	5	6	1^8	-2^9	
18	8	4	3	4	0^9	-1^8	C2, C4, C9
	9	5	4	5	0^8	-1^9	
18	8	5	4	3	2^3	-1^{14}	
	9	6	3	6	0^{14}	-3^3	C8 for m=3
20	4	1	0	1	0^{15}	-1^4	C1, C6(i) for 2-(5,2,1), C9
	15	12	11	12	0^4	-1^{15}	C6(ii) for 2-(5,4,3)
20	7	4	3	2	2^4	-1^{15}	C6(ii) for 2-(5,2,1), C7 for pg(4,2,2)
	12	9	6	9	0^{15}	-3^4	C6(i) for 2-(5,4,3), C9
20	8	4	2	4	0^{15}	-2^4	C8 for m=2, C9
	11	7	6	6	1^4	-1^{15}	
20	9	5	4	4	1^5	-1^{14}	C3
	10	6	4	6	0^{14}	-2^5	C8 for m=2
21	6	2	1	2	0^{14}	-1^6	C6(i) for 2-(7,3,1), C9
	14	10	9	10	0^6	-1^{14}	
21	8	4	3	3	1^6	-1^{14}	C6(ii) for 2-(7,3,1), C7 for pg(3,3,3)
	12	8	6	8	0^{14}	-2^6	C9
22	9	6	3	4	1^{11}	-2^{10}	?
	12	9	6	7	1^{10}	-2^{11}	?
22	10	5	4	5	0^{11}	-1^{10}	C4, C9
	11	6	5	6	0^{10}	-1^{11}	
24	5	2	1	1	1^9	-1^{14}	C3
	18	15	13	15	0^{14}	-2^9	
24	6	2	0	2	0^{20}	-2^3	C6(i) for 2-(4,2,2), C8 for m=2, C9
	17	13	12	12	1^3	-1^{20}	C6(ii) for 2-(4,3,4)
24	7	3	2	2	1^8	-1^{15}	C3
	16	12	10	12	0^{15}	-2^8	C8 for m=2
24	8	4	0	4	0^{21}	-4^2	C6(i) for 2-(3,2,4), C8 for m=2, C9
	15	11	10	8	3^2	-1^{21}	C6(ii) for 2-(3,2,4)
24	8	3	2	3	0^{15}	-1^8	Jørgensen [1211]
	15	10	9	10	0^8	-1^{15}	
24	9	7	2	4	1^{15}	-3^8	Jørgensen [1211]
	14	12	8	8	2^8	-2^{15}	
24	10	5	3	5	0^{18}	-2^5	?
	13	8	7	7	1^5	-1^{18}	?
24	10	8	4	4	2^9	-2^{14}	Jørgensen [1211]
	13	11	6	8	1^{14}	-3^9	
24	11	6	5	5	1^6	-1^{17}	Hobart & Shaw [1110], C3
	12	7	5	7	0^{17}	-2^6	
24	11	7	6	4	3^3	-1^{20}	C6(ii) for 2-(4,2,2)
	12	8	4	8	0^{20}	-4^3	C6(i) for 2-(4,3,4), C8 for m=2, C9

v	k	t	λ	μ	r^f	s^g	comments
24	11	8	7	3	5^2	-1^{21}	
	12	9	3	9	0^{21}	-6^2	C8 for m=3
25	9	6	5	2	4^3	-1^{21}	
	15	12	7	12	0^{21}	-5^3	does not exist by N2
25	10	6	1	6	0^{22}	-5^2	does not exist by N1
	14	10	9	6	4^2	-1^{22}	
26	11	7	4	5	1^{13}	-2^{12}	Jørgensen [1211]
	14	10	7	8	1^{12}	-2^{13}	
26	12	6	5	6	0^{13}	-1^{12}	C2, C4, C9
	13	7	6	7	0^{12}	-1^{13}	
27	7	4	1	2	1^{15}	-2^{11}	?
	19	16	13	14	1^{11}	-2^{15}	?
27	8	4	3	2	2^6	-1^{20}	C7 for pg(3,3,2)
	18	14	11	14	0^{20}	-3^6	
27	9	4	1	4	0^{23}	-3^3	does not exist by N2
	17	12	11	10	2^3	-1^{23}	
27	10	6	3	4	1^{14}	-2^{12}	?
	16	12	9	10	1^{12}	-2^{14}	?
27	11	6	5	4	2^5	-1^{21}	?
	15	10	7	10	0^{21}	-3^5	?
27	12	8	2	8	0^{24}	-6^2	does not exist by N1
	14	10	9	5	5^2	-1^{24}	
28	6	3	2	1	2^7	-1^{20}	does not exist (Brouwer, unpub.)
	21	18	15	18	0^{20}	-3^7	
28	7	2	1	2	0^{20}	-1^7	?
	20	15	14	15	0^7	-1^{20}	?
28	12	6	4	6	0^{21}	-2^6	C6(i) for 2-(7,4,2), C8 for m=2, C9
	15	9	8	8	1^6	-1^{21}	C6(ii) for 2-(7,4,2)
28	13	7	6	6	1^7	-1^{20}	C3
	14	8	6	8	0^{20}	-2^7	C8 for m=2
30	5	1	0	1	0^{24}	-1^5	C1, C6(i) for 2-(6,2,1), C9
	24	20	19	20	0^5	-1^{24}	C6(ii) for 2-(6,5,4)
30	7	5	0	2	1^{20}	-3^9	does not exist (Jørgensen [1211])
	22	20	16	16	2^9	-2^{20}	
30	9	3	2	3	0^{20}	-1^9	C9
	20	14	13	14	0^9	-1^{20}	
30	9	5	4	2	3^5	-1^{24}	C6(ii) for 2-(6,2,1), C7 for pg(5,2,2)
	20	16	12	16	0^{24}	-4^5	C6(i) for 2-(6,5,4), C8 for m=2, C9
30	10	4	2	4	0^{24}	-2^5	C6(i) for 2-(6,3,2), C8 for m=2, C9
	19	13	12	12	1^5	-1^{24}	
30	10	5	0	5	0^{27}	-5^2	C6(i) for 2-(3,2,5), C8 for m=5, C9
	19	14	13	10	4^2	-1^{27}	C6(ii) for 2-(3,2,5)
30	11	5	4	4	1^9	-1^{20}	
	18	12	10	12	0^{20}	-2^9	C8 for m=2, C9, C12 from srg(10,3,0,1)
30	11	9	2	5	1^{21}	-4^8	?
	18	16	11	10	3^8	-2^{21}	?
30	12	6	3	6	0^{25}	-3^4	C6(i) for 2-(5,3,3), C8 for m=3, C9
	17	11	10	9	2^4	-1^{25}	C6(ii) for 2-(5,3,3)
30	12	11	4	5	2^{15}	-3^{14}	?
	17	16	9	10	2^{14}	-3^{15}	?
30	13	8	5	6	1^{15}	-2^{14}	?
	16	11	8	9	1^{14}	-2^{15}	?
30	13	11	6	5	3^9	-2^{20}	?
	16	14	7	10	1^{20}	-4^9	?
30	14	7	6	7	0^{15}	-1^{14}	C4, C9
	15	8	7	8	0^{14}	-1^{15}	
30	14	8	7	6	2^5	-1^{24}	C6(ii) for 2-(6,3,2)
	15	9	6	9	0^{24}	-3^5	C8 for m=3, C9

v	k	t	λ	μ	r^f	s^g	comments
30	14	9	8	5	4^3	-1^{26}	
	15	10	5	10	0^{26}	-5^3	C8 for m=5
32	6	5	1	1	2^{14}	-2^{17}	does not exist (Jørgensen [1210])
	25	24	19	21	1^{17}	-3^{14}	
32	7	4	3	1	3^6	-1^{25}	C5 for GQ(1,3)
	24	21	17	21	0^{25}	-4^6	
32	9	6	1	3	1^{21}	-3^{10}	?
	22	19	15	15	2^{10}	-2^{21}	?
32	10	7	3	3	2^{13}	-2^{18}	?
	21	18	13	15	1^{18}	-3^{13}	?
32	11	6	5	3	3^5	-1^{26}	?
	20	15	11	15	0^{26}	-4^5	?
32	12	6	2	6	0^{28}	-4^3	does not exist by N2
	19	13	12	10	3^3	-1^{28}	
32	13	9	4	6	1^{20}	-3^{11}	?
	18	14	10	10	2^{11}	-2^{20}	?
32	14	7	5	7	0^{24}	-2^7	?
	17	10	9	9	1^7	-1^{24}	?
32	14	10	6	6	2^{12}	-2^{19}	C15
	17	13	8	10	1^{19}	-3^{12}	
32	15	8	7	7	1^8	-1^{23}	C3
	16	9	7	9	0^{23}	-2^8	
32	15	9	8	6	3^4	-1^{27}	
	16	10	6	10	0^{27}	-4^4	C8 for m=2
32	15	11	10	4	7^2	-1^{29}	
	16	12	4	12	0^{29}	-8^2	C8 for m=2

11.5 See Also

add to see also

§VII.10	Strongly regular graphs.

References Cited: [761, 809, 810, 904, 910, 1022, 1109, 1110, 1208, 1209, 1210, 1211, 1306, 1307]

12 Two-Graphs

Edward Spence

12.1 Definitions and Examples

12.1 A *two-graph* (V, Δ) consists of a set V whose elements are the vertices, and a collection Δ of unordered triples of the vertices (the *odd triples*), such that each 4-tuple of V contains, as subsets, an even number of elements of Δ. If (V, Δ) is a two-graph, then so also is the pair $(V, V^3 \setminus \Delta)$; the *complement* of (V, Δ).

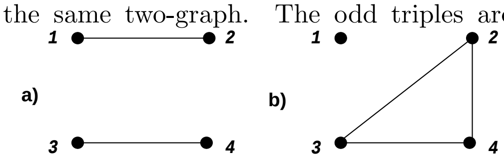
A *clique* of a two-graph (V, Δ) is a subset C of V such that every triple from C is in Δ.

The two-graphs (V, Δ) and (V', Δ') are *isomorphic* if there is a bijection $V \to V'$ that induces a bijection $\Delta \to \Delta'$.

The *automorphism group* $\mathrm{Aut}(V, \Delta)$ of a two-graph (V, Δ) is the group of permutations of V that preserve Δ.

12.2 **Remark** Given a (simple) graph $\Gamma = (V, E)$, the set Δ of triples of V whose induced subgraph in Γ contains an odd number of edges gives rise to a two-graph (V, Δ). In fact, every two-graph can be represented in this way.

12.3 Example The ladder graph on four vertices and a triangle extended by an isolated vertex give rise to the same two-graph. The odd triples are $\{123, 124, 134, 234\}$.



12.4 Remarks

1. In Example 12.3, (b) is obtained from (a) by *switching* the adjacencies of vertex 2. In general, switching with respect to a subset $X \subset V$ consists of interchanging adjacency and nonadjacency between X and its complement $V \setminus X$.
2. Graphs $\Gamma_1 = (V, E_1)$ and $\Gamma_2 = (V, E_2)$ represent the same two-graph if they are related by (the equivalence relation of) switching with respect to some $X \subset V$. Thus a two-graph can be identified with the switching class of a graph.

12.5 The *switching class* of a graph Γ comprises those graphs with ∓ 1 adjacency matrices DAD, where D is a diagonal matrix with entries ± 1, and A is the ∓ 1 adjacency matrix of Γ (-1 for adjacency).

12.6 Remark Since A and DAD (D as above) have the same eigenvalues, the *eigenvalues of a two-graph* are the eigenvalues of the ∓ 1 adjacency matrix of any graph in its switching class.

12.7 Theorem [1507] Two-graphs, switching classes of graphs and the isomorphism classes of eulerian graphs (every vertex has even valency) on n vertices are equal in number.

12.8 Theorem [1856] Let $\Gamma = (V, E)$ be a graph in the switching class of a two-graph (V, Δ). Then $\mathrm{Aut}(\Gamma)$ is a subgroup of $\mathrm{Aut}(V, \Delta)$.

12.9 Theorem [1856] Let $\Gamma_i = (V, E_i)$, $i = 1, 2, \ldots, s$ denote the nonisomorphic graphs in the switching class of a two-graph (V, Δ), and let $|V| = n$, $|\mathrm{Aut}(V, \Delta)| = m$, $|\mathrm{Aut}\Gamma_i| = m_i$. Then $\sum_{i=1}^{s} \frac{m}{m_i} = 2^{n-1}$.

12.10 Remark Mallows and Sloane [1507] prove that every automorphism of a two-graph is also an automorphism of some graph in the switching class of the two-graph. However, there are two-graphs (V, Δ) each of whose switching classes contain graphs, all of which have automorphism group a *proper* subgroup of $\mathrm{Aut}(V, \Delta)$. All such two-graphs on fewer than $n \le 10$ vertices are enumerated in [408].

12.11 Table The numbers $N_T(n)$ of pairwise nonisomorphic two-graphs on n vertices ($n \le 10$). All two-graphs on $n \le 9$ vertices are given in [408], together with their eigenvalues.

n	3	4	5	6	7	8	9	10
$N_T(n)$	2	3	7	16	54	243	2038	33120

12.2 Two-Graphs from Trees

12.12 Construction (Cameron [420])

1. Let T be a tree with edge set E and let Δ be the set of 3-subsets of E not contained in paths in T. Then (E, Δ) is a two-graph.
2. Let X be the end vertices of a reduced vertex-2-colored (black and white) tree T (no divalent vertices) and let Δ comprise the triples of X whose vertices are pairwise joined by paths that meet in a black vertex. Then (X, Δ) is a two-graph. Use of the color white yields the complementary two-graph.

12.13 A two-graph Ω contains a graph Γ as an *induced substructure* if Γ is a subgraph of some graph in the switching class of Ω.

12.14 Theorem [420] A two-graph arises from a tree by Construction 12.12(1) if and only if the two-graph contains neither the pentagon nor the hexagon as an induced substructure. Only isomorphic trees yield isomorphic two-graphs.

12.15 Theorem [420] A two-graph arises from a tree by Construction 12.12(2) if and only if the two-graph does not contain the pentagon as an induced substructure. Only isomorphic colored reduced trees yield isomorphic two-graphs.

12.16 Remark Ishihara [1170] gives an alternative proof to Cameron's Theorem 12.15.

12.17 Remark Cameron [421] produces formulae for the numbers of labeled and labeled reduced two-graphs obtained by Construction 12.12. His proofs use various enumeration results for trees.

12.3 Groups Represented by Two-Graphs

12.18 Let $\Gamma = (V, E)$ be a graph with vertex set V. *The Coxeter group* of Γ, $\mathrm{Cox}(\Gamma)$ has a generator s_v for each $v \in V$, and defining relations $s_v^2 = 1, (s_v s_w)^3 = 1$ if $v \sim w$, and $(s_v s_w)^2 = 1$ if $v \not\sim w$. The set of all products of generators of even length forms a normal subgroup of index 2, denoted by $\mathrm{Cox}^+(\Gamma)$ [611]. Analogously [1861], the *Tsaranov group* of Γ has generators $t_v, v \in V$ and is given by

$$\mathrm{Ts}(\Gamma) := \langle t_v, v \in V \ : \ t_v^3 = 1,\ (t_v t_w^{-1})^2 = 1 \text{ if } v \sim w,\ (t_v t_w)^2 = 1 \text{ if } v \not\sim w \rangle.$$

12.19 Remark Switching Γ with respect to a set of vertices corresponds to replacing Tsaranov's generators for these vertices by their inverses. It follows that the Tsaranov group is an invariant of the two-graph (V, Δ) corresponding to the switching class of Γ. Thus isomorphic two-graphs have isomorphic Tsaranov groups.

12.20 Theorem [1861] The group $\mathrm{Ts}(\Delta)$ of the two-graph (V, Δ) is finite if and only if its smallest eigenvalue is greater than -3. If this is the case, then $\mathrm{Ts}(\Delta) \cong \mathrm{Cox}^+(\Pi)$, where Π is a "Coxeter-Dynkin graph" of type A, D, or E.

12.21 Remark All two-graphs with smallest eigenvalue greater than -3 are identified in [1861].

12.22 Theorem [1861] Let $\Delta(T)$ be the two-graph arising from a tree T by means of Construction 12.12(1). Then $\mathrm{Ts}(\Delta(T)) \cong \mathrm{Cox}^+(T)$.

12.4 Regular Two-Graphs

12.23 Theorem Suppose the graph $\Gamma = (V, E)$ on n vertices has ∓ 1 adjacency matrix A with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_d$ and $-\rho$ of multiplicity $n - d$. Then using the values of $\mathrm{trace}(A)$ and $\mathrm{trace}(A^2)$,

$$\rho^2(n-d) \le d(n-1) \text{ with equality if and only if } \lambda_1 = \lambda_2 = \ldots = \lambda_d.$$

12.24 Equality in Theorem 12.23 is of particular interest for the two-graph with Γ in its switching class. The following two definitions are equivalent [1856]:

1. A two-graph (V, Δ) is *regular* if each pair of vertices in V is contained in the same number of triples of Δ.
2. A two-graph (V, Δ) is *regular* if and only if it has two distinct eigenvalues $\rho_1 > 0 > \rho_2$, say, where $\rho_1\rho_2 = 1 - |V|$. The set $\{\rho_1, \rho_2\}$ is the *spectrum* of the two-graph.

12.25 Remark The switching class of a graph Γ, and hence any two-graph, can be represented geometrically as a set of *equiangular lines* [2066]. Let $-\rho = \lambda_{\min}$ be the smallest eigenvalue of the ∓ 1 adjacency matrix A of Γ (on n vertices) and suppose that, as above, $-\rho$ has multiplicity $n - d$. Then $\rho I + A$ is positive semi-definite of rank d and so can be represented as the Gram matrix of the inner products of n vectors in euclidean d-space. Since these vectors have the same norm $\sqrt{\rho}$ and mutual inner products ± 1, any pair of the n lines spanned by them have the same angle ϕ with $\cos\phi = 1/\rho$. Conversely, given a set of n nonorthogonal equiangular lines in euclidean d-space there exists a two-graph from which it can be constructed by the above method [1856]. Thus, from Theorem 12.23, the maximum cardinality n of such a set satisfies $n \le d(\rho^2 - 1)/(\rho^2 - d)$ and this bound is achieved if and only if the corresponding two-graph is regular.

12.26 Remark In a regular two-graph with n vertices, the eigenvalues ρ_1 and ρ_2 are odd integers except possibly when $\rho_1 + \rho_2 = 0$. In this latter case the ∓ 1 adjacency matrix A of any graph in the switching class of the regular two-graph satisfies $A^2 = (n-1)I$ (the conference two-graph [1856]).

12.27 Theorem [1856] Suppose that (V, Δ) is a regular two-graph with spectrum $\{\rho_1, \rho_2\}$ and let $\Gamma = (V, E)$ be any graph in its switching class with an isolated vertex $v \in V$. Then $(V \setminus \{v\}, E)$ is a strongly regular graph (§VII.10) with Seidel spectrum (the eigenvalues of the ∓ 1 adjacency matrix) $\{\rho_0, \rho_1, \rho_2\}$, where $\rho_0 = \rho_1 + \rho_2$. This graph is a *descendant* of the two-graph.

12.28 Remark In the opposite direction, if Γ is a strongly regular graph (see §VII.10) with parameters (v, k, λ, μ), then the graph obtained from Γ by adjoining an isolated vertex belongs to the switching class of a regular two-graph if and only if $k = 2\mu$ [1856].

12.29 Theorem [1856] Given ρ_2 there are only finitely many possible ρ_1 such that there exists a regular two-graph with spectrum $\{\rho_1, \rho_2\}$ [1856].

12.30 Table The only possible spectra for regular two-graphs with $\rho_2 = -3$ or -5.

$-\rho_2$	3	3	3	3	3	5	5	5	5	5	5	5	5	5	5
ρ_1	1	3	5	9	21	1	3	5	7	15	19	25	35	55	115
n	4	10	16	28	64	6	16	26	36	76	96	126	176	276	576

12.31 Remark Of the pairs (ρ_1, ρ_2) listed in Table 12.30 it is known that $(21, -3)$ and $(115, -5)$ are impossible [1856] while the only ones remaining open are $(15, -5)$ and $(19, -5)$.

▷ Conference Two-Graphs

12.32 A *C-matrix* of order n is a square matrix C with zero diagonal and off-diagonal entries ± 1, such that $C^2 = (n-1)I$. C-matrices are essentially symmetric or skew according as $n \equiv 2$ or $0 \pmod 4$ [673]. Symmetric C-matrices (the ones of interest here) are *conference* matrices.

12.33 Remark Conference matrices are discussed in §V.2.

12.34 Theorem A conference matrix of order n is in the switching class of a regular two-graph with eigenvalues $\rho_1 = \sqrt{n-1}, \rho_2 = -\sqrt{n-1}$. The two-graph is a *conference* two-graph.

12.35 Theorem [1856] Necessary conditions for the existence of a conference two-graph with eigenvalues ρ_1, ρ_2 such that $\rho_1 + \rho_2 = 0$ are that $\rho_1^2 \equiv 1 \pmod 4$, and ρ_1^2 is a sum of squares of integers.

12.36 Theorem [1701] If $q \equiv 1 \pmod 4$ is a prime power, then there exists a conference two-graph of order $q+1$ (the Paley two-graph).

12.37 Theorem [2057] If a conference two-graph of order $m+1$ exists, then there also exists a conference two-graph of order m^d+1 for all $d \geq 1$.

12.38 Theorem [1525] If $q = 4t-1$ is a prime power and $p+1 = 4t+2$ is the order of a conference two-graph, then there exists a conference two-graph on pq^2+1 vertices.

12.39 Remark There are two further infinite families of conference two-graphs, known as the Möbius and Minkowski two-graphs, which are constructed in the setting of Möbius and Minkowski geometry (see [1860, 2115]). The Möbius two-graphs are isomorphic to the Paley two-graphs.

12.40 Construction (*Minkowski two-graphs* [1856]) Let $V = V(2,q)$ be the vector space of dimension 2 over $\mathbb{F}_q$, where q is an odd prime power. Suppose that the one-dimensional subspaces of V are partitioned into two disjoint sets P and N of equal size $(q+1)/2$. Then $(V \cup \{\infty\}, \Delta)$, where Δ comprises the set of triples $\{\infty, x, y\}$, with $x, y \in V$, such that $\langle x-y \rangle \in P$, is a self-complementary conference two-graph on q^2+1 vertices.

12.41 Remark The switching class of a two-graph from Construction 12.40 contains strongly regular graphs of degrees $q(q-1)/2$ and $q(q+1)/2$.

12.42 Theorem [1856] If there exist s mutually orthogonal latin squares of order $2s-1$ or there exists a 2-$(2s^2-2s+1, s, 1)$ design, then there exists a conference two-graph on $(2s-1)^2+1$ vertices.

▷ Symplectic and Orthogonal Two-Graphs

12.43

1. [1856] Let $V = V(2m,2)$ denote the vector space of dimension $2m$ over $\mathbb{F}_2$ and let $B : V \times V \to \mathbb{F}_2$ be a nondegenerate, alternating bilinear form. The *symplectic* two-graph $\Sigma(2m,2) = (V, \Delta)$ consists of the set V and the set $\Delta = \{\{u,v,w\} \subseteq V \ : \ u,v,w \text{ are distinct and } B(u,v)+B(v,w)+B(w,u) = 0\}$. Its eigenvalues are $\rho_1 = 1+2^m, \rho_2 = 1-2^m$.
2. [1856] There are essentially two quadratic forms on $V(2m,2)$ defined by $Q^+(x) = x_1x_2 + x_3x_4 + \cdots + x_{2m-1}x_{2m}$ and $Q^-(x) = x_1^2 + x_2^2 + x_1x_2 + \cdots + x_{2m-1}x_{2m}$. The *orthogonal* two-graph $\Omega^\epsilon(2m,2)$, $\epsilon = \pm$, consists of the set of points $\Omega^\epsilon := \{x \in V \mid Q^\epsilon(x) = 0\}$, and the set of triples of distinct $u,v,w \in \Omega^\epsilon$ such that $B(u,v)+B(v,w)+B(w,u) = 0$. $\Omega^+(2m,2)$ has eigenvalues $\rho_1 = 1+2^{m-1}, \rho_2 = 1-2^m$ and $\Omega^-(2m,2)$ has eigenvalues $\rho_1 = 1+2^m, \rho_2 = 1-2^{m-1}$.

12.44 Remark Taylor [1987] proved that the automorphism group of $\Omega^\epsilon(2m,2)$ is the symplectic group, while $\Sigma(2m,2)$ has automorphism group the symplectic group extended by the translations of $V(2m,2)$. See §VII.8.

▷ Unitary Two-Graphs

12.45 Let H be a nondegenerate hermitian form on the projective plane $\mathrm{PG}(2, q^2)$, where q is an odd prime power, and let $U = \{x \mid H(x,x) = 0\}$ be the corresponding unital. Then $| U |= q^3 + 1$. The *unitary* two-graph $U(q) = (U, \Delta)$ consists of the set Δ of triples $\{x, y, z\}$, with $x, y, z \in U$, for which $H(x, y)H(y, z)H(z, x)$ is a square or nonsquare according as $q \equiv \mp 1 \pmod 4$. The Unitary group $\mathrm{P\Gamma U}(3, q^2)$ acts 2-transitively on Ω.

12.46 Proposition The eigenvalues of $U(q)$ are $\rho_1 = q^2, \rho_2 = -q$ [1987]. Its switching class contains a strongly regular graph of degree $q(q^2 + 1)/2$.

12.47 Remark Taylor [1986] proved that the automorphism group of $U(5)$ is $P\Gamma U(3, 5^2)$.

▷ Two Sporadic Regular Two-Graphs

12.48 Construction There is a unique 4-design with parameters 4-$(23, 7, 1)$ (it is a Witt design). Its residual design, a 3-$(22, 7, 4)$ design, is also unique. In each of these two designs distinct blocks meet in one or three points. The residual design has 176 blocks and its point-block incidence matrix M satisfies $MM^t = 40I + 16J$, $MJ = 56J$, $JM = 7J$. The eigenvalues of MM^t are clearly 392 and 40 and hence M^tM has eigenvalues $392, 40, 0$. Thus $A := M^tM - 5I - 2J$, a symmetric matrix of order 176 with zero diagonal and elements ± 1 elsewhere, has eigenvalues 35 and -5. It is therefore the adjacency matrix of a graph in the switching class of a regular two-graph on 176 vertices.

12.49 Construction If N is the point-block incidence matrix of the 4-$(23, 7, 1)$ design, then $NN^t = 56I + 21J$, $NJ = 77J$, $JN = 7J$. N^tN has eigenvalues $539, 56, 0$ and those of $B := N^tN - 5I - 2J$ are $28, 51, -5$. Hence B satisfies $(B - 51I)(B + 5I) = -3J$, $BJ = 28J$, $NB = 51N - 7J$. A straightforward calculation shows that the symmetric matrix C of order 276, with zero diagonal and ± 1 elsewhere, given by $C = \begin{bmatrix} J - I & J - 2N \\ J - 2N^t & B \end{bmatrix}$ satisfies $(C-55I)(C+5I) = 0$, and hence is in the switching class of a regular two-graph on 276 vertices. This two-graph is unique [917].

▷ Symmetric Hadamard Matrices of Order 36

12.50 Remark Let E be the adjacency matrix of any graph in the switching class of a regular two-graph on 36 vertices. By complementation, if necessary, it may be assumed that the two-graph, and hence E, has eigenvalues 5 and -7, so that $(E - 5I)(E + 7I) = 0$. If $H := E + I$, then H is a symmetric Hadamard matrix of order 36 with constant diagonal (see §V.1). In [408] there were constructed 91 such regular two-graphs, 11 from latin squares of order 6 and 80 from Steiner triple systems of order 15 (see also [916]). A further 136 were found using a computer and a backtracking algorithm [1930]. Subsequently, it was shown in [1565] that the $91+136 = 227$ regular two-graphs found in [408] and [1930] comprise the complete set. Many of these new two-graphs, and some of the older ones, have a particular shape. The corresponding Hadamard matrices contain, as a principal sub-matrix, the matrix A of order 20 defined by $A = \begin{bmatrix} I + P & I - P \\ I - P & I + P \end{bmatrix}$, where P is the ∓ 1 adjacency matrix of the Petersen graph. Assume this to be the case. Then for $\begin{bmatrix} A & N \\ N^t & B \end{bmatrix}$ to be a Hadamard matrix of order 36 it is necessary that $N = \begin{bmatrix} M \\ M \end{bmatrix}$, where M is of size 10×16 and is determined by

Theorem 12.51.

12.51 Theorem [1930] There exist diagonal ±1 matrices D_1, D_2 such that $M = D_1 \begin{bmatrix} C & \mathbf{j} \end{bmatrix} D_2$, where C is the ±1 incidence matrix of a 2-$(10,4,2)$ design (Table II.1.27). Conversely, let C be the ±1 incidence matrix of a 2-$(10,4,2)$ design and suppose that D_1 and D_2 are diagonal matrices of order 10, 16 respectively. If $M = D_1 \begin{bmatrix} C & \mathbf{j} \end{bmatrix} D_2$, and $B = 6I - \frac{1}{2}M^tM$, then $B - I$ is in the switching class of a regular two-graph of size 16,

$$H = \begin{bmatrix} I+P & I-P & M \\ I-P & I+P & M \\ M^t & M^t & B \end{bmatrix},$$

a symmetric Hadamard matrix of order 36 and $H - I$ is in the switching class of a regular two-graph on 36 vertices.

12.52 Remark There are three nonisomorphic 2-$(10,4,2)$ designs and each of them yields regular two-graphs on 36 vertices. In total 100 nonisomorphic regular two-graphs of order 36 are characterized by this method. See [1930] for the details.

12.53 Remark Grishukhin [966] investigated two-graphs from the even unimodular lattice $E_8 \oplus E_8$ and produces a further characterization of these 100 two-graphs.

▷ Completely Regular Two-Graphs

12.54 A regular two-graph is *completely regular* if, for all k, every k-clique is in a constant number α_k of $(k+1)$-cliques.

12.55 Theorem There are unique completely regular two-graphs on 10, 16, 28, 36, and 276 vertices. No others exist except possibly on 1128 or 3160 vertices.

12.56 Remark Neumaier [1659] allowed the further possibilities of 96, 288, and 640 vertices. These were shown subsequently not to exist [290, 294].

▷ Two-Graph Geometries

12.57 Theorem In a regular two-graph with eigenvalues $\rho_1 > \rho_2$, the maximum size of a clique c is given in terms of the smaller eigenvalue ρ_2 by $|c| \le 1 - \rho_2$ [1858].

12.58 A *maximal* clique is one in which this upper bound of Theorem 12.57 is achieved.

12.59 A *two-graph geometry* is a set C of maximal cliques in a regular two-graph (V, Δ), such that each triple from Δ is in a unique clique from C.

12.60 Examples [998]

1. In a regular two-graph, every odd triple is contained in $1 - \frac{1}{4}(\rho_1 + 3)(\rho_2 + 3)$ 4-cliques. When $\rho_2 = -3$, this number equals 1, and these cliques are maximal. Thus by taking C to be the set of all 4-cliques, three two-graph geometries are obtained corresponding to the three regular two-graphs on 10, 16, and 28 vertices.
2. The known regular two-graph on 176 vertices has the Higman-Sims group acting as a 2-transitive group of automorphisms [1987] (see §VII.8). The subgroup M_{22} has an orbit C of size 18480 acting on the maximal 6-cliques of the two-graph and every odd triple is contained in exactly one 6-clique.

▷ **Two-Graphs and Geometry**

12.61 Remark Gunawardena and Moorhouse [987], associate a two-graph with an ovoid in finite orthogonal spaces of type $O_{2n+1}(q)$ (assuming one exists), and show that it is regular. Gunawardena supplies an alternative proof [986].

▷ **Numerical Data**

12.62 Table Data concerning the known regular two-graphs with eigenvalues ρ_1 and $\rho_2 = -3, -5$ and the conference two-graphs on ≤ 50 vertices. N is the number of known nonisomorphic two-graphs on n vertices and N^* indicates that no further two-graphs on n vertices can exist. The main reference is [408]. For $n = 30, 36$ and 38 see also [1930] and [1565]. In the cases $n = 46, 50$ additional work by Mathon [1528] improved $N(46)$ to 97 and $N(50)$ to 19. A further 24 regular two-graphs on 50 vertices were found as a result of the discovery of 12 new Steiner systems $S(2, 4, 25)$ [1335] (see Theorem 12.42), and seven more were recently found [1931], thus increasing $N(50)$ to 50. The case $n = 126$ corresponds to $U(5)$ and $n = 176, 276$ are the two sporadic two-graphs described earlier in Constructions 12.48 and 12.49.

n	6	10	14	16	18	26	28	30
ρ_1, ρ_2	$\pm\sqrt{5}$	± 3	$\pm\sqrt{13}$	$5, -3$	$\pm\sqrt{17}$	± 5	$9, -3$	$\pm\sqrt{29}$
N	1^*	1^*	1^*	1^*	1^*	4^*	1^*	6^*

n	36	38	42	46	50	126	176	276
ρ_1, ρ_2	$7, -5$	$\pm\sqrt{37}$	$\pm\sqrt{41}$	$\pm\sqrt{45}$	± 7	$25, -5$	$35, -5$	$55, -5$
N	227^*	191	18	97	50	1	1	1^*

12.63 Remark It can happen that the switching class of a regular two-graph itself contains strongly regular graphs. There are five values of $n \leq 50$ for which this is possible, namely $n = 10, 16, 26, 36$, and 50. The numbers of such graphs are listed, together with the numbers of nonisomorphic descendants. Here $|(n, \rho_0, \rho_1, \rho_2)|$ denotes the number of strongly regular graphs on n vertices with Seidel spectrum ρ_0, ρ_1, ρ_2.

n	10	16	26	36	50
ρ_1, ρ_2	$(3, -3)$	$(5, -3)$	$(5, -5)$	$(7, -5)$	$(7, -7)$
$\lvert(n-1, \rho_1+\rho_2, \rho_1, \rho_2)\rvert$	1	1	15	3854	312
$\lvert(n, \rho_1, \rho_1, \rho_2)\rvert$	1	1	10	180	1482
$\lvert(n, \rho_2, \rho_1, \rho_2)\rvert$	1	2	10	32548	1482

12.5 See Also

§II.4	t-designs are used in Constructions 12.48 and 12.49.
§III.3	MOLS are used in Theorem 12.42.
§V.1	Hadamard matrices are used to construct regular two-graphs (e.g., in Remark 12.50.
§V.6	Conference matrices are used to construct conference two-graphs (Theorem 12.34).
§VII.10	Strongly regular graphs arise in the switching classes of some two-graphs (e.g., Remark 12.41).

References Cited: [290, 294, 408, 420, 421, 611, 673, 916, 917, 966, 986, 987, 998, 1170, 1335, 1507, 1525, 1528, 1565, 1659, 1701, 1856, 1858, 1860, 1861, 1930, 1931, 1986, 1987, 2057, 2066, 2115]